

# Reachability problems for a wave-wave system with a memory term

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## Abstract

We solve the reachability problem for a coupled wave-wave system with an integro-differential term. The control functions act on one side of the boundary. The estimates on the time is given in terms of the parameters of the problem and they are explicitly computed thanks to Ingham type results. Nevertheless some restrictions appear in our main results. The Hilbert Uniqueness Method is briefly recalled. Our findings can be applied to concrete examples in viscoelasticity theory.

**Keywords:** boundary observability, reachability, Fourier series, hyperbolic integro-differential systems, abstract linear evolution equations

## 1 Introduction

The linear viscoelasticity theory has been extensively studied by many authors, that proposed several mathematical models based on experimental data to tackle such subject. A possible approach relies on the following physical assumption: the present stress is given by a functional of the past history of the deformation gradient. Such functionals can be represented by means of convolution integrals. This leads to wave equations in which a so-called memory term also appears, see the seminal papers of Dafermos [4, 5] and [30, 14]. In this framework an important issue is to identify suitable class of integral kernels that match with the physical models. For example, decreasing exponential kernels arise in the analysis of Maxwell fluids or Poynting -Thomson solids, see e.g. [29, 31]. It is also noteworthy to mention that such kernels satisfy the principle of fading memory, *the memory of a simple material fades in time*, introduced in [3].

Our aim, justified by the previous remarks, is to investigate the reachability for a system constituted of a wave equation with a memory term and another wave equation coupled by lower order terms. Precisely, given  $a, b \in \mathbb{R}$  we consider the following system

$$\begin{cases} u_{1tt}(t, x) - u_{1xx}(t, x) + \beta \int_0^t e^{-\eta(t-s)} u_{xx}(s, x) ds + au_2(t, x) = 0, & (0 < \beta < \eta) \\ u_{2tt}(t, x) - u_{2xx}(t, x) + bu_1(t, x) = 0, \end{cases} \quad t \in (0, T), \quad x \in (0, \pi) \quad (1)$$

subject to the boundary conditions

$$u_1(t, 0) = u_2(t, 0) = 0, \quad u_1(t, \pi) = g_1(t), \quad u_2(t, \pi) = g_2(t) \quad t \in (0, T), \quad (2)$$

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and with null initial conditions

$$u_i(0, x) = u_{it}(0, x) = 0 \quad x \in (0, \pi), \quad i = 1, 2. \quad (3)$$

We wish to solve a reachability problem for (1) of the following type: given  $T > 0$  and taking  $(u_i^0, u_i^1)$ ,  $i = 1, 2$ , whose regularity we will specify later, one has to find  $g_i \in L^2(0, T)$ ,  $i = 1, 2$  such that the weak solution  $u$  of problem (1)-(3) satisfies the final conditions

$$u_i(T, x) = u_i^0(x), \quad u_{it}(T, x) = u_i^1(x), \quad x \in (0, \pi), \quad i = 1, 2. \quad (4)$$

In the literature coupled wave-wave equations were investigated by studying boundary stabilization, see [9]. The exact synchronization for a coupled system of wave equations with Dirichlet boundary conditions was successfully treated by Li and Rao [17]. They studied the  $n$ -dimensional case when the coupling matrix is very general. However, their method does not allow to get precise estimates on the controllability time.

In [1] F. Alabau-Boussouira considered a system where the coupling parameters are all equal, obtaining an observability inequality for small coupling parameter and large time  $T$  and then, by duality, an exact indirect controllability result.

In this paper we solve the reachability problems for the coupled wave-wave with an integro-differential term by the HUM method, see [18, 19, 20] and by means of non-harmonic analysis techniques. In this framework Ingham type estimates, see [8], play an important role. We already used this approach to study the reachability for one equation, see [22, 23] and to treat the case of a wave-Petrovsky system with a memory term, see [24]. For a different class of integral kernels see [21] and for the hidden regularity in the case of general kernels see [25].

However the estimates obtained do not include the case wave-wave without memory as limit case as  $\beta \rightarrow 0^+$

$$\begin{cases} u_{1tt} - u_{1xx} + au_2 = 0 \\ u_{2tt} - u_{2xx} + bu_1 = 0 \end{cases} \quad \text{on } (0, T) \times (0, \pi), \quad (5)$$

because, as formulas (50) and (51) clearly show, the eigenvectors of the integro-differential operator are not bounded as  $\beta \rightarrow 0^+$ .

The method is based on a representation formula for the solution  $(u_1, u_2)$ , established in Section 4

$$\begin{aligned} u_1(t) &= \sum_{n=1}^{\infty} \left( C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right), \\ u_2(t) &= \sum_{n=1}^{\infty} \left( d_n D_n e^{i\zeta_n t} + \overline{d_n} \overline{D_n} e^{-i\overline{\zeta_n} t} + c_n C_n e^{i\omega_n t} + \overline{c_n} \overline{C_n} e^{-i\overline{\omega_n} t} \right) + \mathcal{E} e^{-\eta t}, \end{aligned}$$

where

$$|\mathcal{E}|^2 \leq M \sum_{n=1}^{\infty} \left( |C_n|^2 + |d_n D_n|^2 \right), \quad (M > 0).$$

We will prove the following reachability result (see Theorem 6.1) where we will give an estimate of the control time.

**Theorem 1.1** *Let  $\beta < 1/2$ . For any  $T > \frac{2\pi}{\sqrt{1-4\beta^2}}$  and  $(u_i^0, u_i^1) \in L^2(0, \pi) \times H^{-1}(0, \pi)$ ,  $i = 1, 2$ , there exist  $g_i \in L^2(0, T)$ ,  $i = 1, 2$ , such that the weak solution  $(u_1, u_2)$  of system*

$$\begin{cases} u_{1tt}(t, x) - u_{1xx}(t, x) + \beta \int_0^t e^{-\eta(t-s)} u_{1xx}(s, x) ds + au_2(t, x) = 0, \\ u_{2tt}(t, x) - u_{2xx}(t, x) + bu_1(t, x) = 0, \end{cases} \quad t \in (0, T), \quad x \in (0, \pi) \quad (6)$$

with boundary conditions

$$u_1(t, 0) = u_2(t, 0) = 0, \quad u_1(t, \pi) = g_1(t), \quad u_2(t, \pi) = g_2(t) \quad t \in (0, T), \quad (7)$$

and null initial values

$$u_i(0, x) = u_{it}(0, x) = 0 \quad x \in (0, \pi), \quad i = 1, 2, \quad (8)$$

verifies the final conditions

$$u_i(T, x) = u_i^0(x), \quad u_{it}(T, x) = u_i^1(x), \quad x \in (0, \pi), \quad i = 1, 2. \quad (9)$$

Due to the duality between controllability and observability we will first prove Ingham type inequalities (see Theorem 5.16).

**Theorem 1.2** *Let  $\{\omega_n\}_{n \in \mathbb{N}}$ ,  $\{r_n\}_{n \in \mathbb{N}}$  and  $\{\zeta_n\}_{n \in \mathbb{N}}$  be sequences of pairwise distinct numbers such that  $\omega_n \neq \zeta_m$ ,  $\omega_n \neq \overline{\zeta_m}$ ,  $r_n \neq i\omega_m$ ,  $r_n \neq i\zeta_m$ ,  $r_n \neq -\eta$ ,  $\zeta_n \neq 0$ , for any  $n, m \in \mathbb{N}$ . Assume that there exist  $\gamma > 0$ ,  $\alpha, \chi \in \mathbb{R}$ ,  $n' \in \mathbb{N}$ ,  $\mu > 0$ ,  $\nu > 1/2$ , such that*

$$\liminf_{n \rightarrow \infty} (\Re \omega_{n+1} - \Re \omega_n) = \liminf_{n \rightarrow \infty} (\Re \zeta_{n+1} - \Re \zeta_n) = \gamma,$$

$$\lim_{n \rightarrow \infty} \Im \omega_n = \alpha > 0,$$

$$\lim_{n \rightarrow \infty} r_n = \chi < 0,$$

$$\lim_{n \rightarrow \infty} \Im \zeta_n = 0,$$

$$|d_n| \asymp |\zeta_n|, \quad |c_n| \leq \frac{M}{|\omega_n|},$$

$$|R_n| \leq \frac{\mu}{n^\nu} \left( |C_n|^2 + |d_n D_n|^2 \right)^{1/2} \quad \forall n \geq n', \quad |R_n| \leq \mu \left( |C_n|^2 + |d_n D_n|^2 \right)^{1/2} \quad \forall n \leq n'.$$

Then, for  $\gamma > 4\alpha$  and  $T > \frac{2\pi}{\sqrt{\gamma^2 - 16\alpha^2}}$  we have

$$\int_0^T (|u_1(t)|^2 + |u_2(t)|^2) dt \asymp \sum_{n=1}^{\infty} \left( |C_n|^2 + |d_n D_n|^2 \right). \quad (10)$$

The observability time may be improved making an extra assumption on the initial data. Indeed, if we assume the condition  $|C_n| \leq M|d_n D_n|$  on the coefficients of the series instead of  $\gamma > 4\alpha$ , then we can make use of Theorem 5.10 instead of Theorem 5.9, obtaining the observability estimates for  $T > \frac{2\pi}{\gamma}$  (see Theorem 5.17).

**Theorem 1.3** *Let assume the hypotheses of Theorem 1.2 and the condition*

$$|C_n| \leq M|d_n D_n|. \quad (11)$$

Then, for  $T > \frac{2\pi}{\gamma}$  we have

$$\int_0^T (|u_1(t)|^2 + |u_2(t)|^2) dt \asymp \sum_{n=1}^{\infty} \left( |C_n|^2 + |d_n D_n|^2 \right). \quad (12)$$

The plan of our paper is the following. In Section 2 we give some preliminary results. In Section 3 we describe the Hilbert Uniqueness Method. In Section 4 we carry out a detailed spectral analysis to give a representation formula for the solution of the wave-wave coupled system with memory. In Section 5 we prove the observability estimates. Finally, in Section 6 we give a reachability result for the coupled system with memory.

## 2 Preliminaries

Throughout the paper, we will adopt the convention to write  $F \asymp G$  if there exist two positive constants  $c_1$  and  $c_2$  such that  $c_1 F \leq G \leq c_2 F$ .

Let  $X$  be a real Hilbert space with scalar product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|$ . For any  $T \in (0, \infty]$  we denote by  $L^1(0, T; X)$  the usual spaces of measurable functions  $v : (0, T) \rightarrow X$  such that one has

$$\|v\|_{1,T} := \int_0^T \|v(t)\| dt < \infty.$$

We shall use the shorter notation  $\|v\|_1$  for  $\|v\|_{1,\infty}$ . We denote by  $L_{loc}^1(0, \infty; X)$  the space of functions belonging to  $L^1(0, T; X)$  for any  $T \in (0, \infty)$ . In the case of  $X = \mathbb{R}$ , we will use the abbreviations  $L^1(0, T)$  and  $L_{loc}^1(0, \infty)$  to denote the spaces  $L^1(0, T; \mathbb{R})$  and  $L_{loc}^1(0, \infty; \mathbb{R})$ , respectively.

Classical results for integral equations (see, e.g., [6, Theorem 2.3.5]) ensure that, for any kernel  $k \in L_{loc}^1(0, \infty)$  and  $\psi \in L_{loc}^1(0, \infty; X)$ , the problem

$$\varphi(t) - k * \varphi(t) = \psi(t), \quad t \geq 0, \quad (13)$$

admits a unique solution  $\varphi \in L_{loc}^1(0, \infty; X)$ . In particular, if we take  $\psi = k$  in (13), we can consider the unique solution  $\varrho_k \in L_{loc}^1(0, \infty)$  of

$$\varrho_k(t) - k * \varrho_k(t) = k(t), \quad t \geq 0.$$

Such a solution is called the *resolvent kernel* of  $k$ . Furthermore, for any  $\psi$  the solution  $\varphi$  of (13) is given by the variation of constants formula

$$\varphi(t) = \psi(t) + \varrho_k * \psi(t), \quad t \geq 0,$$

where  $\varrho_k$  is the resolvent kernel of  $k$ .

We recall some results concerning integral equations in case of decreasing exponential kernels, see for example [23, Corollary 2.2].

**Proposition 2.1** *For  $0 < \beta < \eta$  and  $T > 0$  the following properties hold true.*

- (i) *The resolvent kernel of  $k(t) = \beta e^{-\eta t}$  is  $\varrho_k(t) = \beta e^{(\beta-\eta)t}$ .*
- (ii) *Given  $\psi \in L_{loc}^1(-\infty, T; X)$ , a function  $\varphi \in L_{loc}^1(-\infty, T; X)$  is a solution of*

$$\varphi(t) - \beta \int_t^T e^{-\eta(s-t)} \varphi(s) ds = \psi(t) \quad t \leq T,$$

*if and only if*

$$\varphi(t) = \psi(t) + \beta \int_t^T e^{(\beta-\eta)(s-t)} \psi(s) ds \quad t \leq T.$$

*Moreover, there exist two positive constants  $c_1, c_2$  depending on  $\beta, \eta, T$  such that*

$$c_1 \int_0^T |\varphi(t)|^2 dt \leq \int_0^T |\psi(t)|^2 dt \leq c_2 \int_0^T |\varphi(t)|^2 dt. \quad (14)$$

We state and prove a result, that will allow us to give an equivalent way to write the solution of our problem.

**Lemma 2.2** *Given  $\lambda, \beta, \eta \in \mathbb{R}$ ,  $a \in \mathbb{R} \setminus \{0\}$  and  $b \in \mathbb{R}$ , a couple  $(f, g)$  of scalar functions defined on the interval  $[0, \infty)$  is a solution of the system*

$$\begin{cases} f'' + \lambda f - \lambda \beta \int_0^t e^{-\eta(t-s)} f(s) ds + ag = 0, \\ g'' + \lambda g + bf = 0, \end{cases} \quad t \geq 0, \quad (15)$$

*if and only if  $f$  is a solution of the equation*

$$f^{(5)} + \eta f^{(4)} + 2\lambda f''' + \lambda(2\eta - \beta)f'' + (\lambda^2 - ab)f' + (\lambda^2(\eta - \beta) - \eta ab)f = 0, \quad t \geq 0, \quad (16)$$

*the condition*

$$f^{(4)}(0) = -2\lambda f''(0) + \lambda \beta f'(0) + (ab - \eta \lambda \beta - \lambda^2)f(0) \quad (17)$$

*is satisfied and  $g$  is given by*

$$g = -\frac{1}{a} \left( f'' + \lambda f - \lambda \beta \int_0^t e^{-\eta(t-s)} f(s) ds \right). \quad (18)$$

*Proof.* Let  $(f, g)$  be a solution of (15). Differentiating the first equation in (15), we get

$$f''' + \lambda f' + \eta \lambda \beta \int_0^t e^{-\eta(t-s)} f(s) ds - \lambda \beta f + ag' = 0, \quad (19)$$

whence

$$ag'(0) = -f'''(0) - \lambda f'(0) + \lambda \beta f(0). \quad (20)$$

Substituting in (19) the identity

$$\lambda \beta \int_0^t e^{-\eta(t-s)} f(s) ds = f'' + \lambda f + ag,$$

we obtain

$$f''' + \eta f'' + \lambda f' + \lambda(\eta - \beta)f + ag' + \eta ag = 0. \quad (21)$$

Differentiating yet again, we have

$$f^{(4)} + \eta f''' + \lambda f'' + \lambda(\eta - \beta)f' + ag'' + \eta ag' = 0,$$

whence, by using the second equation in (15), that is  $ag'' = -abf - \lambda ag$ , we get

$$f^{(4)} + \eta f''' + \lambda f'' + \lambda(\eta - \beta)f' - abf + \eta ag' - \lambda ag = 0. \quad (22)$$

Thanks to (20) and  $ag(0) = -f''(0) - \lambda f(0)$ , we have

$$\begin{aligned} f^{(4)}(0) &= -\eta f'''(0) - \lambda f''(0) - \lambda(\eta - \beta)f'(0) + abf(0) - \eta ag'(0) + \lambda ag(0) \\ &= -\eta f'''(0) - \lambda f''(0) - \lambda(\eta - \beta)f'(0) + abf(0) + \eta f'''(0) \\ &\quad + \eta \lambda f'(0) - \eta \lambda \beta f(0) - \lambda f''(0) - \lambda^2 f(0) \\ &= -2\lambda f''(0) + \lambda \beta f'(0) + (ab - \eta \lambda \beta - \lambda^2)f(0), \end{aligned}$$

so formula (17) for  $f^{(4)}(0)$  holds true. Moreover, by differentiating (22) we obtain

$$f^{(5)} + \eta f^{(4)} + \lambda f''' + \lambda(\eta - \beta)f'' - abf' + \eta ag'' - \lambda ag' = 0.$$

By using again  $g'' = -bf - \lambda g$  we get

$$f^{(5)} + \eta f^{(4)} + \lambda f''' + \lambda(\eta - \beta)f'' - abf' - \eta abf - \lambda ag' - \eta \lambda ag = 0.$$

From (21) it follows

$$-ag' - \eta ag = f''' + \eta f'' + \lambda f' + \lambda(\eta - \beta)f,$$

and hence we have

$$f^{(5)} + \eta f^{(4)} + 2\lambda f''' + \lambda(2\eta - \beta)f'' + (\lambda^2 - ab)f' + (\lambda^2(\eta - \beta) - \eta ab)f = 0,$$

that is  $f$  is a solution of the differential equation (16). Finally, from the first equation in (15) we deduce that  $g$  is given by (18).

Conversely, if  $f$  satisfies (16) – (17), multiplying the differential equation by  $e^{\eta t}$  and integrating from 0 to  $t$ , we obtain

$$\begin{aligned} \int_0^t e^{\eta s} f^{(5)}(s) ds + \eta \int_0^t e^{\eta s} f^{(4)}(s) ds + 2\lambda \int_0^t e^{\eta s} f'''(s) ds + 2\eta\lambda \int_0^t e^{\eta s} f''(s) ds \\ - \lambda\beta \int_0^t e^{\eta s} f''(s) ds + (\lambda^2 - ab) \int_0^t e^{\eta s} f'(s) ds + (\lambda^2(\eta - \beta) - \eta ab) \int_0^t e^{\eta s} f(s) ds = 0. \end{aligned}$$

Integrating by parts the first, the third, the fifth and the sixth integral, we have

$$\begin{aligned} e^{\eta t} f^{(4)} - f^{(4)}(0) + 2\lambda e^{\eta t} f'' - 2\lambda f''(0) - \lambda\beta e^{\eta t} f' + \lambda\beta f'(0) + \eta\lambda\beta e^{\eta t} f \\ - \eta\lambda\beta f(0) - \eta^2\lambda\beta \int_0^t e^{\eta s} f(s) ds + (\lambda^2 - ab)e^{\eta t} f - (\lambda^2 - ab)f(0) - \lambda^2\beta \int_0^t e^{\eta s} f(s) ds = 0. \end{aligned}$$

Using the condition (17) and multiplying by  $e^{-\eta t}$ , we obtain

$$\begin{aligned} f^{(4)} + 2\lambda f'' - \lambda\beta f' + \eta\lambda\beta f - \eta^2\lambda\beta \int_0^t e^{-\eta(t-s)} f(s) ds \\ + (\lambda^2 - ab)f - \lambda^2\beta \int_0^t e^{-\eta(t-s)} f(s) ds = 0. \quad (23) \end{aligned}$$

Moreover, by (18) it follows

$$ag' = -f''' - \lambda f' + \lambda\beta f - \eta\lambda\beta \int_0^t e^{-\eta(t-s)} f(s) ds,$$

and hence

$$ag'' = -f^{(4)} - \lambda f'' + \lambda\beta f' - \eta\lambda\beta f + \eta^2\lambda\beta \int_0^t e^{-\eta(t-s)} f(s) ds.$$

Therefore, thanks to the previous identity and (23) we have

$$ag'' = \lambda f'' + (\lambda^2 - ab)f - \lambda^2\beta \int_0^t e^{-\eta(t-s)} f(s) ds,$$

whence, in view of (18) we get

$$ag'' = -\lambda ag - abf.$$

Finally, by (18) and the above equation, it follows that the couple  $(f, g)$  is a solution of the system (15).

□

The following lemma is analogous to that of [23, Lemma 2.3]. For the reader's convenience we prefer to state and prove it the same.

**Lemma 2.3** *Given  $\lambda, \beta, \eta \in \mathbb{R}$  and  $h \in C(\mathbb{R})$ , if  $g \in C^3(\mathbb{R})$  is a solution of the third order differential equation*

$$g''' + \eta g'' + \lambda g' + \lambda(\eta - \beta)g = h \quad \text{in } \mathbb{R}, \quad (24)$$

*then  $g$  is also a solution of the integro-differential equation*

$$g'' + \lambda g - \lambda\beta \int_0^t e^{-\eta(t-s)} g(s) ds = e^{-\eta t}(g''(0) + \lambda g(0)) + \int_0^t e^{-\eta(t-s)} h(s) ds \quad t \in \mathbb{R}. \quad (25)$$

*Proof.* Multiplying the differential equation (24) by  $e^{\eta t}$  and integrating from 0 to  $t$ , we obtain

$$\int_0^t e^{\eta s} g'''(s) ds + \eta \int_0^t e^{\eta s} g''(s) ds + \lambda \int_0^t e^{\eta s} g'(s) ds + \lambda(\eta - \beta) \int_0^t e^{\eta s} g(s) ds = \int_0^t e^{\eta s} h(s) ds.$$

Integrating by parts the first term and the third one, we have

$$e^{\eta t} g'' - g''(0) + \lambda e^{\eta t} g - \lambda g(0) - \lambda\beta \int_0^t e^{\eta s} g(s) ds = \int_0^t e^{\eta s} h(s) ds.$$

Finally, if we multiply by  $e^{-\eta t}$ , then we obtain (25).  $\square$

### 3 The Hilbert Uniqueness Method

For reader's convenience, in this section we will describe the Hilbert Uniqueness Method for coupled wave equations with a memory term. For another approach based on the ontoness of the solution operator, see e.g. [13, 33].

Given  $k \in L_{loc}^1(0, \infty)$  and  $a, b \in \mathbb{R}$ , we consider the following coupled system:

$$\begin{cases} u_{1tt}(t, x) - u_{1xx}(t, x) + \int_0^t k(t-s) u_{1xx}(s, x) ds + a u_2(t, x) = 0, \\ u_{2tt}(t, x) - u_{2xx}(t, x) + b u_1(t, x) = 0, \end{cases} \quad t \in (0, T), \quad x \in (0, \pi) \quad (26)$$

subject to the boundary conditions

$$u_1(t, 0) = u_2(t, 0) = 0, \quad u_1(t, \pi) = g_1(t), \quad u_2(t, \pi) = g_2(t) \quad t \in (0, T), \quad (27)$$

and with null initial conditions

$$u_i(0, x) = u_{it}(0, x) = 0 \quad x \in (0, \pi), \quad i = 1, 2. \quad (28)$$

For a reachability problem we mean the following: given  $T > 0$  and taking  $(u_i^0, u_i^1)$ ,  $i = 1, 2$ , in a suitable space, that we will introduce later, find  $g_i \in L^2(0, T)$ ,  $i = 1, 2$  such that the weak solution  $u$  of problem (26)-(28) satisfies the final conditions

$$u_i(T, x) = u_i^0(x), \quad u_{it}(T, x) = u_i^1(x), \quad x \in (0, \pi), \quad i = 1, 2. \quad (29)$$

One can solve such reachability problems by the HUM method. To see that, we proceed as follows.

Given  $(z_i^0, z_i^1) \in (C_c^\infty(0, \pi))^2$ ,  $i = 1, 2$ , we introduce the *adjoint* system of (26), that is

$$\begin{cases} z_{1tt}(t, x) - z_{1xx}(t, x) + \int_t^T k(s-t)z_{1xx}(s, x)ds + bz_2(t, x) = 0, \\ z_{2tt}(t, x) - z_{2xx}(t, x) + az_1(t, x) = 0, \\ z_i(t, 0) = z_i(t, \pi) = 0 \quad t \in [0, T], \quad i = 1, 2, \end{cases} \quad t \in (0, T), \quad x \in (0, \pi) \quad (30)$$

with final data

$$z_i(T, \cdot) = z_i^0, \quad z_{it}(T, \cdot) = z_i^1, \quad i = 1, 2. \quad (31)$$

The above problem is well-posed, see e.g. [29]. Thanks to the regularity of the final data, the solution  $(z_1, z_2)$  of (30)–(31) is regular enough to consider the nonhomogeneous problem

$$\begin{cases} \varphi_{1tt}(t, x) - \varphi_{1xx}(t, x) + \int_0^t k(t-s)\varphi_{1xx}(s, x)ds + a\varphi_2(t, x) = 0 \\ \varphi_{2tt}(t, x) - \varphi_{2xx}(t, x) + b\varphi_1(t, x) = 0 \\ \varphi_i(0, x) = \varphi_{it}(0, x) = 0 \quad x \in (0, \pi), \quad i = 1, 2, \\ \varphi_1(t, 0) = 0, \quad \varphi_1(t, \pi) = z_{1x}(t, \pi) - \int_t^T k(s-t)z_{1x}(s, \pi)ds \\ \varphi_2(t, 0) = 0, \quad \varphi_2(t, \pi) = z_{2x}(t, \pi). \end{cases} \quad t \in (0, T), \quad x \in (0, \pi), \quad t \in [0, T], \quad (32)$$

As in the non-integral case, it can be proved that problem (32) admits a unique solution  $(\varphi_1, \varphi_2)$ . So, we can introduce the following linear operator: for any  $(z_i^0, z_i^1) \in (C_c^\infty(0, \pi))^2$ ,  $i = 1, 2$ , we define

$$\Psi(z_1^0, z_1^1, z_2^0, z_2^1) = (-\varphi_{1t}(T, \cdot), \varphi_1(T, \cdot), -\varphi_{2t}(T, \cdot), \varphi_2(T, \cdot)). \quad (33)$$

For any  $(\xi_i^0, \xi_i^1) \in (C_c^\infty(0, \pi))^2$ ,  $i = 1, 2$ , let  $(\xi_1, \xi_2)$  be the solution of

$$\begin{cases} \xi_{1tt}(t, x) - \xi_{1xx}(t, x) + \int_t^T k(s-t)\xi_{1xx}(s, x)ds + b\xi_2(t, x) = 0 \\ \xi_{2tt}(t, x) - \xi_{2xx}(t, x) + a\xi_1(t, x) = 0 \\ \xi_i(t, 0) = \xi_i(t, \pi) = 0 \quad t \in [0, T], \\ \xi_i(T, \cdot) = \xi_i^0, \quad \xi_{it}(T, \cdot) = \xi_i^1. \end{cases} \quad t \in (0, T), \quad x \in (0, \pi), \quad i = 1, 2, \quad (34)$$

We will prove that

$$\begin{aligned} & \langle \Psi(z_1^0, z_1^1, z_2^0, z_2^1), (\xi_1^0, \xi_1^1, \xi_2^0, \xi_2^1) \rangle_{L^2} \\ &= \int_0^T \varphi_1(t, \pi) \left( \xi_{1x}(t, \pi) - \int_t^T k(s-t) \xi_{1x}(s, \pi) ds \right) dt + \int_0^T \varphi_2(t, \pi) \xi_{2x}(t, \pi) dt. \end{aligned} \quad (35)$$

To this end, we multiply the first equation in (32) by  $\xi_1$  and integrate on  $[0, T] \times [0, \pi]$ , so we have

$$\begin{aligned} & \int_0^\pi \int_0^T \varphi_{1tt}(t, x) \xi_1(t, x) dt dx - \int_0^\pi \int_0^T \varphi_{1xx}(t, x) \xi_1(t, x) dx dt \\ &+ \int_0^\pi \int_0^T \int_0^t k(t-s) \varphi_{1xx}(s, x) ds \xi_1(t, x) dt dx + a \int_0^T \int_0^\pi \varphi_2(t, x) \xi_1(t, x) dx dt = 0. \end{aligned}$$



If we take into account that

$$\int_0^T \int_0^t k(t-s) \varphi_{1xx}(s, x) ds \xi_1(t, x) dt = \int_0^T \varphi_{1xx}(s, x) \int_s^T k(t-s) \xi_1(t, x) dt ds$$

and integrate by parts, then we have

$$\begin{aligned} & \int_0^\pi (\varphi_{1t}(T, x) \xi_1^0(x) - \varphi_1(T, x) \xi_1^1(x)) dx + \int_0^\pi \int_0^T \varphi_1(t, x) \xi_{1tt}(t, x) dt dx \\ & \quad + \int_0^T \varphi_1(t, \pi) \xi_{1x}(t, \pi) dt - \int_0^T \int_0^\pi \varphi_1(t, x) \xi_{1xx}(t, x) dx dt \\ & \quad - \int_0^T \varphi_1(s, \pi) \int_s^T k(t-s) \xi_{1x}(t, \pi) dt ds + \int_0^\pi \int_0^T \varphi_1(s, x) \int_s^T k(t-s) \xi_{1xx}(t, x) dt ds dx \\ & \quad + a \int_0^T \int_0^\pi \varphi_2(t, x) \xi_1(t, x) dx dt = 0. \end{aligned}$$

As a consequence of the above equation and

$$\xi_{1tt} - \xi_{1xx} + \int_t^T k(s-t) \xi_{1xx}(s, \cdot) ds = -b \xi_2,$$

we obtain

$$\begin{aligned} & \int_0^\pi (\varphi_{1t}(T, x) \xi_1^0(x) - \varphi_1(T, x) \xi_1^1(x)) dx + \int_0^T \varphi_1(t, \pi) \left( \xi_{1x}(t, \pi) - \int_t^T k(s-t) \xi_{1x}(s, \pi) ds \right) dt \\ & \quad + \int_0^T \int_0^\pi (a \varphi_2(t, x) \xi_1(t, x) - b \varphi_1(t, x) \xi_2(t, x)) dx dt = 0. \quad (36) \end{aligned}$$

In a similar way, we multiply the second equation in (32) by  $\xi_2$  and integrate by parts on  $[0, T] \times [0, \pi]$  to get

$$\begin{aligned} & \int_0^\pi (\varphi_{2t}(T, x) \xi_2^0(x) - \varphi_2(T, x) \xi_2^1(x)) dx + \int_0^\pi \int_0^T \varphi_2(t, x) \xi_{2tt}(t, x) dt dx \\ & \quad + \int_0^T \varphi_2(t, \pi) \xi_{2x}(t, \pi) dt - \int_0^T \int_0^\pi \varphi_2(t, x) \xi_{2xx}(t, x) dx dt + b \int_0^T \int_0^\pi \varphi_1(t, x) \xi_2(t, x) dx dt = 0, \end{aligned}$$

whence, in virtue of

$$\xi_{2tt} - \xi_{2xx} = -a \xi_1,$$

we get

$$\begin{aligned} & \int_0^\pi (\varphi_{2t}(T, x) \xi_2^0(x) - \varphi_2(T, x) \xi_2^1(x)) dx + \int_0^T \varphi_2(t, \pi) \xi_{2x}(t, \pi) dt \\ & \quad + \int_0^T \int_0^\pi (b \varphi_1(t, x) \xi_2(t, x) - a \varphi_2(t, x) \xi_1(t, x)) dx dt = 0. \quad (37) \end{aligned}$$

If we sum equations (36) and (37), then we have

$$\begin{aligned} & \langle \Psi(z_1^0, z_1^1, z_2^0, z_2^1), (\xi_1^0, \xi_1^1, \xi_2^0, \xi_2^1) \rangle_{L^2} \\ & \quad = \int_0^\pi (-\varphi_{1t}(T, x) \xi_1^0(x) + \varphi_1(T, x) \xi_1^1(x) - \varphi_{2t}(T, x) \xi_1^0(x) + \varphi_2(T, x) \xi_1^1(x)) dx \\ & \quad = \int_0^T \varphi_1(t, \pi) \left( \xi_{1x}(t, \pi) - \int_t^T k(s-t) \xi_{1x}(s, \pi) ds \right) dt + \int_0^T \varphi_2(t, \pi) \xi_{2x}(t, \pi) dt, \quad (38) \end{aligned}$$

that is, (35) holds true.

Taking  $\xi_i^0 = z_i^0$  and  $\xi_i^1 = z_i^1$ ,  $i = 1, 2$ , in (35) yields

$$\begin{aligned} \langle \Psi(z_1^0, z_1^1, z_2^0, z_2^1), (z_1^0, z_1^1, z_2^0, z_2^1) \rangle_{L^2} \\ = \int_0^T \left| z_{1x}(t, \pi) - \int_t^T k(s-t) z_{1x}(s, \pi) ds \right|^2 dt + \int_0^T |z_{2x}(t, \pi)|^2 dt. \end{aligned} \quad (39)$$

As a consequence, we can introduce a semi-norm on the space  $(C_c^\infty(\Omega))^4$ . Indeed, for  $(z_i^0, z_i^1) \in (C_c^\infty(\Omega))^2$ ,  $i = 1, 2$ , we define

$$\|(z_1^0, z_1^1, z_2^0, z_2^1)\|_F := \left( \int_0^T \left| z_{1x}(t, \pi) - \int_t^T k(s-t) z_{1x}(s, \pi) ds \right|^2 dt + \int_0^T |z_{2x}(t, \pi)|^2 dt \right)^{1/2}. \quad (40)$$

In view of Proposition 2.1,  $\|\cdot\|_F$  is a norm if and only if the following uniqueness theorem holds.

**Theorem 3.1** *If  $(z_1, z_2)$  is the solution of problem (30)–(31) such that*

$$z_{1x}(t, \pi) = z_{2x}(t, \pi) = 0, \quad \forall t \in [0, T],$$

*then*

$$z_1(t, x) = z_2(t, x) = 0 \quad \forall (t, x) \in [0, T] \times [0, \pi].$$

If we are able to establish Theorem 3.1, then we can define the Hilbert space  $F$  as the completion of  $(C_c^\infty(\Omega))^4$  for the norm (40). Moreover, the operator  $\Psi$  extends uniquely to a continuous operator, denoted again by  $\Psi$ , from  $F$  to the dual space  $F'$  in such a way that  $\Psi : F \rightarrow F'$  is an isomorphism.

In conclusion, if we prove Theorem 3.1 and, for example,  $F = (H_0^1(0, \pi) \times L^2(0, \pi))^2$  with the equivalence of the respective norms, then, taking  $(u_i^0, u_i^1) \in L^2(0, \pi) \times H^{-1}(0, \pi)$ ,  $i = 1, 2$ , we can solve the reachability problem (26)–(29).

## 4 Representation of the solution as Fourier series

### 4.1 Spectral analysis

The aim of this section will be to give a complete spectral analysis for the coupled system.

We will recast our system of coupled wave equations with a memory term in an abstract setting. Indeed, we consider a self-adjoint positive linear operator  $L : D(L) \subset H \rightarrow H$  on a Hilbert space  $H$  with dense domain  $D(L)$ . We denote by  $\{\lambda_n\}_{n \geq 1}$  a strictly increasing sequence of eigenvalues for the operator  $L$  with  $\lambda_n > 0$  and  $\lambda_n \rightarrow \infty$  and we assume that the sequence of the corresponding eigenvectors  $\{w_n\}_{n \geq 1}$  constitutes a Hilbert basis for  $H$ .

We fix two real numbers  $a \neq 0$ ,  $b$  and consider the following coupled system:

$$\begin{cases} u_1''(t) + Lu_1(t) - \beta \int_0^t e^{-\eta(t-s)} Lu_1(s) ds + au_2(t) = 0 \\ u_2''(t) + Lu_2(t) + bu_1(t) = 0 \\ u_i(0) = u_i^0, \quad u_i'(0) = u_i^1, \quad i = 1, 2. \end{cases} \quad t \geq 0, \quad (41)$$

If we take the initial data  $(u_i^0, u_i^1)$ ,  $i = 1, 2$ , belonging to  $D(\sqrt{L}) \times H$ , then we can expand them according to the eigenvectors  $w_n$  to obtain:

$$\begin{aligned} u_i^0 &= \sum_{n=1}^{\infty} \alpha_{in} w_n, & \alpha_{in} &= \langle u_i^0, w_n \rangle, & \|u_i^0\|_{D(\sqrt{L})}^2 &:= \sum_{n=1}^{\infty} \alpha_{in}^2 \lambda_n, \\ u_i^1 &= \sum_{n=1}^{\infty} \rho_{in} w_n, & \rho_{in} &= \langle u_i^1, w_n \rangle, & \|u_i^1\|_H^2 &:= \sum_{n=1}^{\infty} \rho_{in}^2. \end{aligned} \quad (42)$$

Our target is to write the components  $u_1, u_2$  of the solution of system (41) as sums of series, that is

$$u_i(t) = \sum_{n=1}^{\infty} f_{in}(t)w_n, \quad f_{in}(t) = \langle u_i(t), w_n \rangle, \quad i = 1, 2.$$

To this end, we put the above expressions for  $u_1$  and  $u_2$  into (41) and multiply by  $w_n$ , so for any  $n \in \mathbb{N}$   $(f_{1n}, f_{2n})$  is the solution of the system

$$\begin{cases} f_{1n}'' + \lambda_n f_{1n} - \beta \lambda_n \int_0^t e^{-\eta(t-s)} f_{1n}(s) ds + a f_{2n} = 0, \\ f_{2n}'' + \lambda_n f_{2n} + b f_{1n} = 0, \\ f_{in}(0) = \alpha_{in}, \quad f_{in}'(0) = \rho_{in}, \quad i = 1, 2. \end{cases} \quad (43)$$

Thanks to lemma 2.2 with  $\lambda = \lambda_n$ ,  $(f_{1n}, f_{2n})$  is the solution of problem (43) if and only if  $f_{1n}$  is the solution of the Cauchy problem

$$\begin{cases} f_{1n}^{(5)} + \eta f_{1n}^{(4)} + 2\lambda_n f_{1n}''' + \lambda_n(2\eta - \beta) f_{1n}'' + (\lambda_n^2 - ab) f_{1n}' + (\lambda_n^2(\eta - \beta) - \eta ab) f_{1n} = 0 & t \geq 0, \\ f_{1n}(0) = \alpha_{1n}, \\ f_{1n}'(0) = \rho_{1n}, \\ f_{1n}''(0) = -\lambda_n \alpha_{1n} - a \alpha_{2n}, \\ f_{1n}'''(0) = -\lambda_n \rho_{1n} + \beta \lambda_n \alpha_{1n} - a \rho_{2n}, \\ f_{1n}^{(4)}(0) = (\lambda_n^2 - \eta \beta \lambda_n + ab) \alpha_{1n} + 2a \lambda_n \alpha_{2n} + \beta \lambda_n \rho_{1n}, \end{cases} \quad (44)$$

and  $f_{2n}$  is given by

$$f_{2n} = -\frac{1}{a} \left( f_{1n}'' + \lambda_n f_{1n} - \beta \lambda_n \int_0^t e^{-\eta(t-s)} f_{1n}(s) ds \right).$$

If we introduce the linear operator  $\Upsilon_n$  defined by

$$\Upsilon_n(v)(t) := -\frac{1}{a} \left( v''(t) + \lambda_n v(t) - \beta \lambda_n \int_0^t e^{-\eta(t-s)} v(s) ds \right) \quad t \geq 0, \quad (45)$$

then  $f_{2n}$  can be written as

$$f_{2n}(t) = \Upsilon_n(f_{1n})(t) \quad t \geq 0. \quad (46)$$

We also note that for any  $z \in \mathbb{C}$

$$\Upsilon_n(e^{zt}) = -\frac{1}{a} \left[ \left( z^2 + \lambda_n - \frac{\beta \lambda_n}{\eta + z} \right) e^{zt} + \frac{\beta \lambda_n}{\eta + z} e^{-\eta t} \right]. \quad (47)$$

## 4.2 The fifth order ordinary differential equation

We proceed to solve the Cauchy problem (44). To this end, we have to evaluate the solutions of the 5<sup>th</sup>-degree characteristic equation in the variable  $Z$

$$Z^5 + \eta Z^4 + 2\lambda_n Z^3 + \lambda_n(2\eta - \beta) Z^2 + (\lambda_n^2 - ab) Z + \lambda_n^2(\eta - \beta) - \eta ab = 0. \quad (48)$$

By means of the singular perturbation theory we get the five solutions of (48): one is a real number  $r_n$  and the other four  $i\omega_n, -i\overline{\omega_n}, i\zeta_n, -i\overline{\zeta_n}$  are pairwise complex conjugate numbers. Moreover,  $r_n, \omega_n$  and  $\zeta_n$  exhibit the following asymptotic behavior as  $n$  tends to  $\infty$ :

$$r_n = \beta - \eta - \frac{\beta(\beta - \eta)^2}{\lambda_n} + O\left(\frac{1}{\lambda_n^2}\right) = \beta - \eta + O\left(\frac{1}{\lambda_n}\right), \quad (49)$$

$$\begin{aligned}\omega_n &= \sqrt{\lambda_n} + \frac{\beta}{2} \left( \frac{3}{4}\beta - \eta \right) \frac{1}{\sqrt{\lambda_n}} + i \left[ \frac{\beta}{2} - \left( \frac{\beta(\beta - \eta)^2}{2} + \frac{ab}{2\beta} \right) \frac{1}{\lambda_n} \right] + O\left(\frac{1}{\lambda_n^{3/2}}\right) \\ &= \sqrt{\lambda_n} + i\frac{\beta}{2} + O\left(\frac{1}{\sqrt{\lambda_n}}\right),\end{aligned}\quad (50)$$

$$\zeta_n = \sqrt{\lambda_n} + \frac{\eta ab}{2\beta\lambda_n^{3/2}} + i \left( \frac{ab}{2\beta\lambda_n} + \frac{a^2b^2}{2\beta^3\lambda_n^2} \right) + O\left(\frac{1}{\lambda_n^{5/2}}\right) = \sqrt{\lambda_n} + i\frac{ab}{2\beta\lambda_n} + O\left(\frac{1}{\lambda_n^{3/2}}\right). \quad (51)$$

Therefore, we are able to write the solution  $f_{1n}(t)$  of (44) in the form

$$f_{1n}(t) = R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t}, \quad (52)$$

where the coefficients  $R_n \in \mathbb{R}$  and  $C_n, D_n \in \mathbb{C}$  are unknown. Since the function  $f_{1n}(t)$  have to satisfy the initial conditions in (44), to determine  $R_n$ ,  $C_n$  and  $D_n$  we will solve the system

$$\begin{cases} R_n + C_n + \overline{C_n} + D_n + \overline{D_n} = f_{1n}(0), \\ r_n R_n + i\omega_n C_n - i\overline{\omega_n} \overline{C_n} + i\zeta_n D_n - i\overline{\zeta_n} \overline{D_n} = f'_{1n}(0), \\ r_n^2 R_n - \omega_n^2 C_n - \overline{\omega_n^2} \overline{C_n} - \zeta_n^2 D_n - \overline{\zeta_n^2} \overline{D_n} = f''_{1n}(0), \\ r_n^3 R_n - i\omega_n^3 C_n + i\overline{\omega_n^3} \overline{C_n} - i\zeta_n^3 D_n + i\overline{\zeta_n^3} \overline{D_n} = f'''_{1n}(0), \\ r_n^4 R_n + \omega_n^4 C_n + \overline{\omega_n^4} \overline{C_n} + \zeta_n^4 D_n + \overline{\zeta_n^4} \overline{D_n} = f^{(4)}_{1n}(0). \end{cases} \quad (53)$$

Indeed, we obtain that the coefficients have the following asymptotic behavior as  $n$  tends to  $\infty$ :

$$R_n = \frac{\beta}{\lambda_n} (\alpha_{1n}(\beta - \eta) + \rho_{1n}) + (\alpha_{1n} + \rho_{1n} + \alpha_{2n} + \rho_{2n}) O\left(\frac{1}{\lambda_n^2}\right), \quad (54)$$

$$\begin{aligned}C_n &= \frac{\alpha_{1n}}{2} - \frac{i}{4\beta} (\beta^2 \alpha_{1n} + 2\beta \rho_{1n} + 2a\alpha_{2n}) \frac{1}{\lambda_n^{1/2}} + \frac{1}{2\beta^2} ((ab - \beta^3(\beta - \eta))\alpha_{1n} \\ &\quad - \beta(\beta^2 \rho_{1n} + \eta a\alpha_{2n}) - \beta a\rho_{2n}) \frac{1}{\lambda_n} + (\alpha_{1n} + \rho_{1n} + \alpha_{2n} + \rho_{2n}) O\left(\frac{1}{\lambda_n^{3/2}}\right)\end{aligned} \quad (55)$$

$$\begin{aligned}D_n &= i\frac{a\alpha_{2n}}{2\beta\lambda_n^{1/2}} + \frac{a}{2\beta^2} (\beta\eta\alpha_{2n} + \beta\rho_{2n} - b\alpha_{1n}) \frac{1}{\lambda_n} + \frac{i}{2\beta^3} (2a^2b\alpha_{2n} - \eta\beta^2 a\rho_{2n} + 2\eta\beta ab\alpha_{1n} + \beta ab\rho_{1n}) \frac{1}{\lambda_n^{3/2}} \\ &\quad + (\alpha_{1n} + \rho_{1n} + \alpha_{2n} + \rho_{2n}) O\left(\frac{1}{\lambda_n^2}\right).\end{aligned} \quad (56)$$

Accordingly, we can write  $f_{1n}(t)$  by means of formula (52), where the coefficients  $R_n$ ,  $C_n$  and  $D_n$  are given by formulas (54)-(56) respectively. Moreover, thanks to (46), we can also get the expression for  $f_{2n}(t)$ , that is

$$f_{2n}(t) = \Upsilon_n (R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t}). \quad (57)$$

We will observe that the function  $f_{2n}(t)$  can be written in a more handleable form. To this end, first we recall the following result (see e.g. [23, Section 6])

**Lemma 4.1** *Approximated solutions of the cubic equation*

$$Z^3 + \eta Z^2 + \lambda_n Z + \lambda_n(\eta - \beta) = 0, \quad (58)$$

are given by

$$r_n = \beta - \eta - \frac{\beta(\beta - \eta)^2}{\lambda_n} + O\left(\frac{1}{\lambda_n^2}\right), \quad (59)$$

$$z_n = -\frac{\beta}{2} + \frac{\beta(\beta - \eta)^2}{2} \frac{1}{\lambda_n} + i \left[ \sqrt{\lambda_n} + \frac{\beta}{2} \left( \frac{3}{4} \beta - \eta \right) \frac{1}{\sqrt{\lambda_n}} \right] + O\left(\frac{1}{\lambda_n^{3/2}}\right). \quad (60)$$

Therefore, comparing (49) with (59), we have that the numbers  $r_n$  are approximated solutions of (58), and hence the function  $t \rightarrow R_n e^{r_n t}$  is a solution of the third order differential equation

$$g''' + \eta g'' + \lambda_n g' + \lambda_n(\eta - \beta)g = 0 \quad \text{in } \mathbb{R}. \quad (61)$$

**Lemma 4.2** *The numbers  $i\omega_n$ , with  $\omega_n$  defined by (50), are approximated solutions of the cubic equation*

$$Z^3 + \eta Z^2 + \lambda_n Z + \lambda_n(\eta - \beta) = -\frac{ab}{\beta}.$$

*Proof.* The comparison of (50) with (60) yields

$$i\omega_n = z_n + \frac{ab}{2\beta\lambda_n}.$$

Since

$$\begin{aligned} & (i\omega_n)^3 + \eta(i\omega_n)^2 + \lambda_n i\omega_n + \lambda_n(\eta - \beta) \\ &= z_n^3 + \eta z_n^2 + \lambda_n z_n + \lambda_n(\eta - \beta) + 3z_n^2 \frac{ab}{2\beta\lambda_n} + 3z_n \frac{a^2 b^2}{4\beta^2 \lambda_n^2} + \frac{a^3 b^3}{8\beta^3 \lambda_n^3} + 2\eta z_n \frac{ab}{2\beta\lambda_n} + \eta \frac{a^2 b^2}{4\beta^2 \lambda_n^2} + \frac{ab}{2\beta}, \end{aligned}$$

and in virtue of Lemma 4.1 we have

$$z_n^3 + \eta z_n^2 + \lambda_n z_n + \lambda_n(\eta - \beta) = 0,$$

then we get

$$(i\omega_n)^3 + \eta(i\omega_n)^2 + \lambda_n i\omega_n + \lambda_n(\eta - \beta) = -\frac{3ab}{2\beta} + \frac{ab}{2\beta} + O\left(\frac{1}{\sqrt{\lambda_n}}\right) = -\frac{ab}{\beta} + O\left(\frac{1}{\sqrt{\lambda_n}}\right).$$

that is, our claim holds true.  $\square$

Thanks to Lemma 4.2, the numbers  $i\omega_n$  and their conjugate numbers  $-i\overline{\omega_n}$  are approximated solutions of the cubic equation

$$Z^3 + \eta Z^2 + \lambda_n Z + \lambda_n(\eta - \beta) = -\frac{ab}{\beta},$$

so, it follows that the function  $t \rightarrow C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}$  is a solution of the third order differential equation

$$g''' + \eta g'' + \lambda_n g' + \lambda_n(\eta - \beta)g = -\frac{ab}{\beta}g \quad \text{in } \mathbb{R}. \quad (62)$$

In virtue of (61) and (62), the function

$$g_n(t) = R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}$$

is a solution of the third order differential equation

$$g''' + \eta g'' + \lambda_n g' + \lambda_n(\eta - \beta)g = -\frac{ab}{\beta}(C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}) \quad \text{in } \mathbb{R}. \quad (63)$$

Therefore, we can apply Lemma 2.3 with  $h(t) = -\frac{ab}{\beta}(C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t})$ : thanks to (25) and (45), we have

$$\Upsilon_n(g_n(t)) = -\frac{1}{a}e^{-\eta t}(g_n''(0) + \lambda_n g_n(0)) + \frac{b}{\beta} \int_0^t e^{-\eta(t-s)}(C_n e^{i\omega_n s} + \overline{C_n} e^{-i\overline{\omega_n} s})ds. \quad (64)$$

From (53) and (44) it follows that

$$\begin{aligned} g_n''(0) &= f_{1n}''(0) + \zeta_n^2 D_n + \overline{\zeta_n^2 D_n} = -\lambda_n \alpha_{1n} - a\alpha_{2n} + \zeta_n^2 D_n + \overline{\zeta_n^2 D_n} \\ \lambda_n g_n(0) &= \lambda_n f_{1n}(0) - \lambda_n D_n - \lambda_n \overline{D_n} = \lambda_n \alpha_{1n} - \lambda_n D_n - \lambda_n \overline{D_n}. \end{aligned}$$

Thanks to (51) we have  $\zeta_n^2 - \lambda_n = O\left(\frac{1}{\sqrt{\lambda_n}}\right)$ , so we see that

$$g_n''(0) + \lambda_n g_n(0) = -a\alpha_{2n} + (\alpha_{1n} + \rho_{1n} + \alpha_{2n} + \rho_{2n})O\left(\frac{1}{\lambda_n}\right).$$

Moreover

$$\int_0^t e^{-\eta(t-s)} e^{i\omega_n s} ds = \frac{1}{\eta + i\omega_n} (e^{i\omega_n t} - e^{-\eta t}).$$

Set

$$c_n = \frac{b}{\beta(\eta + i\omega_n)}, \quad (65)$$

from (64) we obtain

$$\Upsilon_n(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}) = c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} + (\alpha_{2n} - 2\Re(c_n C_n))e^{-\eta t}. \quad (66)$$

Moreover, thanks to (47) we have

$$\Upsilon_n(e^{i\zeta_n t}) = \frac{1}{a} \left( \zeta_n^2 - \lambda_n + \frac{\beta \lambda_n}{\eta + i\zeta_n} \right) e^{i\zeta_n t} - \frac{\beta \lambda_n}{a(\eta + i\zeta_n)} e^{-\eta t}.$$

Therefore, if we define

$$d_n = \frac{1}{a} \left( \zeta_n^2 - \lambda_n + \frac{\beta \lambda_n}{\eta + i\zeta_n} \right), \quad (67)$$

and

$$E_n = \alpha_{2n} - 2\Re(c_n C_n) - \frac{2\beta \lambda_n}{a} \Re\left(\frac{D_n}{\eta + i\zeta_n}\right), \quad (68)$$

thanks to (57) and (66),  $f_{2n}(t)$  can be written in the following form

$$f_{2n}(t) = d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} + c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} + E_n e^{-\eta t}. \quad (69)$$

We also note that

$$|d_n| \asymp |\zeta_n| \asymp \sqrt{\lambda_n}, \quad |c_n| \leq \frac{M}{|\omega_n|}. \quad (70)$$

The proof of the following lemma is straightforward in virtue of (56) and (70), so we omit it.

**Lemma 4.3** *Set*

$$E_n = \alpha_{2n} - 2\Re(c_n C_n) - \frac{2\beta \lambda_n}{a} \Re\left(\frac{D_n}{\eta + i\zeta_n}\right),$$

*there exists a constant  $M > 0$  such that*

$$\left| \sum_{n=1}^{\infty} E_n \right|^2 \leq M \sum_{n=1}^{\infty} (|C_n|^2 + |d_n D_n|^2).$$

Now, we state and prove some properties about the coefficients, that show some differences with respect to the analogous ones in [23, 24].

**Lemma 4.4** *The following statements hold true.*

(i) *For any  $n \in \mathbb{N}$  one has*

$$|C_n|^2 + \lambda_n |D_n|^2 \asymp \frac{1}{\lambda_n} (\alpha_{1n}^2 \lambda_n + \rho_{1n}^2 + \alpha_{2n}^2 \lambda_n + \rho_{2n}^2). \quad (71)$$

(ii) *There exists a constant  $M > 0$  such that for any  $n \in \mathbb{N}$  one has*

$$|R_n| \leq \frac{M}{\lambda_n^{1/2}} \left( |C_n|^2 + \lambda_n |D_n|^2 \right)^{1/2}. \quad (72)$$

*Proof.* (i) From (55) it follows that

$$\begin{aligned} |C_n|^2 &= \frac{1}{4} \alpha_{1n}^2 + \frac{1}{16\beta^2} (\beta^2 \alpha_{1n} + 2\beta \rho_{1n} + 2a\alpha_{2n})^2 \frac{1}{\lambda_n} \\ &\quad + \frac{\alpha_{1n}}{2\beta^2} ((ab - \beta^3(\beta - \eta))\alpha_{1n} - \beta(\beta^2 \rho_{1n} + \eta a\alpha_{2n}) - \beta a\rho_{2n}) \frac{1}{\lambda_n} \\ &\quad + (\alpha_{1n}^2 + \rho_{1n}^2 + \alpha_{2n}^2 + \rho_{2n}^2) O\left(\frac{1}{\lambda_n^2}\right). \end{aligned} \quad (73)$$

Moreover, from (56) we deduce that

$$\begin{aligned} \lambda_n^{1/2} D_n &= i \frac{a\alpha_{2n}}{2\beta} + \frac{a}{2\beta^2} (\beta\eta\alpha_{2n} + \beta\rho_{2n} - b\alpha_{1n}) \frac{1}{\lambda_n^{1/2}} \\ &\quad + \frac{i}{2\beta^3} (2a^2 b\alpha_{2n} + 2\eta\beta ab\alpha_{1n} + \beta ab\rho_{1n} - \eta\beta^2 a\rho_{2n}) \frac{1}{\lambda_n} + (\alpha_{1n} + \rho_{1n} + \alpha_{2n} + \rho_{2n}) O\left(\frac{1}{\lambda_n^{3/2}}\right), \end{aligned}$$

whence

$$\begin{aligned} \lambda_n |D_n|^2 &= \frac{a^2 \alpha_{2n}^2}{4\beta^2} + \frac{a^2}{4\beta^4} (\beta\eta\alpha_{2n} + \beta\rho_{2n} - b\alpha_{1n})^2 \frac{1}{\lambda_n} \\ &\quad + \frac{a\alpha_{2n}}{2\beta^4} (2a^2 b\alpha_{2n} + 2\eta\beta ab\alpha_{1n} + \beta ab\rho_{1n} - \eta\beta^2 a\rho_{2n}) \frac{1}{\lambda_n} + (\alpha_{1n}^2 + \rho_{1n}^2 + \alpha_{2n}^2 + \rho_{2n}^2) O\left(\frac{1}{\lambda_n^2}\right). \end{aligned} \quad (74)$$

Now, putting together (73) and (74), we have

$$\begin{aligned} |C_n|^2 + \lambda_n |D_n|^2 &= \frac{1}{4} \left( \alpha_{1n}^2 + \frac{\rho_{1n}^2}{\lambda_n} + \frac{a^2}{\beta^2} \left( \alpha_{2n}^2 + \frac{\rho_{2n}^2}{\lambda_n} \right) \right) \\ &\quad + \frac{1}{16\beta^2} (\beta^2 \alpha_{1n} + 2a\alpha_{2n})^2 \frac{1}{\lambda_n} + \frac{\rho_{1n}}{4\beta} (\beta^2 \alpha_{1n} + 2a\alpha_{2n}) \frac{1}{\lambda_n} \\ &\quad + \frac{\alpha_{1n}}{2\beta^2} ((ab - \beta^3(\beta - \eta))\alpha_{1n} - \beta(\beta^2 \rho_{1n} + \eta a\alpha_{2n}) - \beta a\rho_{2n}) \frac{1}{\lambda_n} \\ &\quad + \frac{a^2}{4\beta^4} (\beta\eta\alpha_{2n} - b\alpha_{1n})^2 \frac{1}{\lambda_n} + \frac{a^2 \rho_{2n}}{2\beta^3} (\beta\eta\alpha_{2n} - b\alpha_{1n}) \frac{1}{\lambda_n} \\ &\quad + \frac{a\alpha_{2n}}{2\beta^4} (2a^2 b\alpha_{2n} + 2\eta\beta ab\alpha_{1n} + \beta ab\rho_{1n} - \eta\beta^2 a\rho_{2n}) \frac{1}{\lambda_n} + (\alpha_{1n}^2 + \rho_{1n}^2 + \alpha_{2n}^2 + \rho_{2n}^2) O\left(\frac{1}{\lambda_n^2}\right). \end{aligned}$$

We can neglect the indices  $n \in \mathbb{N}$  such that  $\alpha_{1n} = \rho_{1n} = \alpha_{2n} = \rho_{2n} = 0$ , because the present evaluation will be used in summing series. So, we can assume that for any  $n \in \mathbb{N}$   $(\alpha_{1n}, \rho_{1n}, \alpha_{2n}, \rho_{2n}) \neq (0, 0, 0, 0)$ , and hence by the previous formula we obtain

$$\begin{aligned} & \frac{|C_n|^2 + \lambda_n |D_n|^2}{\alpha_{1n}^2 + \frac{\rho_{1n}^2}{\lambda_n} + \frac{a^2}{\beta^2} \left( \alpha_{2n}^2 + \frac{\rho_{2n}^2}{\lambda_n} \right)} \\ &= \frac{1}{4} + \frac{(\alpha_{1n}^2 + (\alpha_{1n} + \alpha_{2n})(\rho_{1n} + \alpha_{2n} + \rho_{2n})) O\left(\frac{1}{\lambda_n}\right)}{\alpha_{1n}^2 + \frac{\rho_{1n}^2}{\lambda_n} + \frac{a^2}{\beta^2} \left( \alpha_{2n}^2 + \frac{\rho_{2n}^2}{\lambda_n} \right)} \rightarrow \frac{1}{4}, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

taking into account, for example, that

$$\frac{\alpha_{1n} \rho_{1n}}{\lambda_n} = \frac{\alpha_{1n}}{\lambda_n^{1/3}} \frac{\rho_{1n}}{\lambda_n^{2/3}} \leq \frac{\alpha_{1n}^2}{\lambda_n^{2/3}} + \frac{\rho_{1n}^2}{\lambda_n^{4/3}}.$$

In conclusion, (71) holds true.

(ii) From (54) we have

$$|R_n|^2 = \frac{\beta^2}{\lambda_n^2} (\alpha_{1n}(\beta - \eta) + \rho_{1n})^2 + (\alpha_{1n} + \rho_{1n})(\alpha_{1n} + \rho_{1n} + \alpha_{2n} + \rho_{2n}) O\left(\frac{1}{\lambda_n^3}\right).$$

Moreover, thanks to (71), there exists a constant  $c > 0$  such that

$$|C_n|^2 + \lambda_n |D_n|^2 \geq \frac{c}{\lambda_n} (\alpha_{1n}^2 \lambda_n + \rho_{1n}^2 + \alpha_{2n}^2 \lambda_n + \rho_{2n}^2).$$

Therefore, from the above formulas we get

$$\frac{|R_n|^2}{|C_n|^2 + \lambda_n |D_n|^2} \leq \frac{1}{c \lambda_n} \frac{\beta^2 (\alpha_{1n}(\beta - \eta) + \rho_{1n})^2 + (\alpha_{1n} + \rho_{1n})(\alpha_{1n} + \rho_{1n} + \alpha_{2n} + \rho_{2n}) O\left(\frac{1}{\lambda_n}\right)}{\alpha_{1n}^2 \lambda_n + \rho_{1n}^2 + \alpha_{2n}^2 \lambda_n + \rho_{2n}^2},$$

that is, (72) follows.  $\square$

In conclusion, taking into account of any result of the present section we have proved the following representation formula for the solution of the coupled system.

**Theorem 4.5** *The solution of problem (41) can be written as series in the following way*

$$\begin{aligned} u_1(t) &= \sum_{n=1}^{\infty} \left( C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right) w_n, \\ u_2(t) &= \sum_{n=1}^{\infty} \left( d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} + c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} + E_n e^{-\eta t} \right) w_n, \end{aligned} \tag{75}$$

where

$$\begin{aligned} r_n &= \beta - \eta + O\left(\frac{1}{\lambda_n}\right), \\ \omega_n &= \sqrt{\lambda_n} + i\frac{\beta}{2} + O\left(\frac{1}{\sqrt{\lambda_n}}\right), \\ \zeta_n &= \sqrt{\lambda_n} + i\frac{ab}{2\beta\lambda_n} + O\left(\frac{1}{\lambda_n^{3/2}}\right), \\ |R_n| &\leq \frac{M}{\lambda_n^{1/2}} \left( |C_n|^2 + |d_n D_n|^2 \right)^{1/2}, \quad \left| \sum_{n=1}^{\infty} E_n \right|^2 \leq M \sum_{n=1}^{\infty} \left( |C_n|^2 + |d_n D_n|^2 \right), \\ |d_n| &\asymp \sqrt{\lambda_n}, \quad |c_n| \leq \frac{M}{\sqrt{\lambda_n}}, \quad (M > 0) \\ \sum_{n=1}^{\infty} \lambda_n \left( |C_n|^2 + |d_n D_n|^2 \right) &\asymp \|u_1^0\|_{D(\sqrt{L})}^2 + \|u_1^1\|_H^2 + \|u_2^0\|_{D(\sqrt{L})}^2 + \|u_2^1\|_H^2. \end{aligned}$$



## 5 Ingham type estimates

Our goal is to prove an inverse inequality and a direct inequality for the pair  $(u_1, u_2)$  defined by

$$\begin{aligned} u_1(t) &= \sum_{n=1}^{\infty} \left( C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right), \\ u_2(t) &= \sum_{n=1}^{\infty} \left( d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} + c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right) + \mathcal{E} e^{-\eta t}, \end{aligned} \quad (76)$$

with  $\omega_n, C_n, \zeta_n, D_n, d_n, c_n \in \mathbb{C}$  and  $r_n, R_n, \mathcal{E} \in \mathbb{R}$ . We will assume that there exist  $\gamma > 0$ ,  $\alpha, \chi \in \mathbb{R}$ ,  $n' \in \mathbb{N}$ ,  $\mu > 0$ ,  $\nu > 1/2$ , such that

$$\liminf_{n \rightarrow \infty} (\Re \omega_{n+1} - \Re \omega_n) = \liminf_{n \rightarrow \infty} (\Re \zeta_{n+1} - \Re \zeta_n) = \gamma, \quad (77)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \Im \omega_n &= \alpha > 0, \\ \lim_{n \rightarrow \infty} r_n &= \chi < 0, \\ \lim_{n \rightarrow \infty} \Im \zeta_n &= 0, \end{aligned} \quad (78)$$

$$|d_n| \asymp |\zeta_n|, \quad |c_n| \leq \frac{M}{|\omega_n|}, \quad (79)$$

$$|R_n| \leq \frac{\mu}{n^\nu} \left( |C_n|^2 + |d_n D_n|^2 \right)^{1/2} \quad \forall n \geq n', \quad |R_n| \leq \mu \left( |C_n|^2 + |d_n D_n|^2 \right)^{1/2} \quad \forall n \leq n'. \quad (80)$$

### 5.1 Outline of the proof

Before to proceed with our computations, we will outline briefly our reasoning. Firstly, to shorten our formulas we introduce the following notations

$$\mathcal{U}_1^C(t) = \sum_{n=1}^{\infty} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}), \quad \mathcal{U}_1^D(t) = \sum_{n=1}^{\infty} (D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t}), \quad \mathcal{U}_1^R(t) = \sum_{n=1}^{\infty} R_n e^{r_n t}, \quad (81)$$

$$\mathcal{U}_2^D(t) = \sum_{n=1}^{\infty} (d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t}), \quad \mathcal{U}_2^C(t) = \sum_{n=1}^{\infty} (c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t}), \quad (82)$$

so we can write the functions  $u_1, u_2$  as

$$u_1 = \mathcal{U}_1^C + \mathcal{U}_1^D + \mathcal{U}_1^R, \quad u_2 - \mathcal{E} e^{-\eta t} = \mathcal{U}_2^D + \mathcal{U}_2^C.$$

If  $k(t)$  is a suitable positive function, see (85) below, our first goal will be to estimate

$$\int_0^\infty k(t) |\mathcal{U}_1^C(t) + \mathcal{U}_1^D(t) + \mathcal{U}_1^R(t)|^2 dt + \int_0^\infty k(t) |\mathcal{U}_2^D(t) + \mathcal{U}_2^C(t)|^2 dt,$$

unless a finite number of terms in the series.

By reason of  $2ab \geq -\frac{1}{2}a^2 - 2b^2$  we have  $|a+b|^2 \geq \frac{1}{2}a^2 - b^2$ , so we can observe that

$$|\mathcal{U}_1^C(t) + \mathcal{U}_1^D(t) + \mathcal{U}_1^R(t)|^2 \geq \frac{1}{2} |\mathcal{U}_1^C(t)|^2 - |\mathcal{U}_1^D(t) + \mathcal{U}_1^R(t)|^2 \geq \frac{1}{2} |\mathcal{U}_1^C(t)|^2 - 2|\mathcal{U}_1^D(t)|^2 - 2|\mathcal{U}_1^R(t)|^2,$$

$$|\mathcal{U}_2^D(t) + \mathcal{U}_2^C(t)|^2 \geq \frac{1}{2} |\mathcal{U}_2^D(t)|^2 - |\mathcal{U}_2^C(t)|^2.$$

Bearing in mind (80), since  $k(t)$  is positive from the above inequalities we can deduce

$$\begin{aligned} & \int_0^\infty k(t) |\mathcal{U}_1^C(t) + \mathcal{U}_1^D(t) + \mathcal{U}_1^R(t)|^2 dt + \int_0^\infty k(t) |\mathcal{U}_2^D(t) + \mathcal{U}_2^C(t)|^2 dt \\ & \geq \int_0^\infty k(t) \left( \frac{1}{2} |\mathcal{U}_1^C(t)|^2 - 2 |\mathcal{U}_1^D(t)|^2 \right) dt + \int_0^\infty k(t) \left( \frac{1}{2} |\mathcal{U}_2^D(t)|^2 - |\mathcal{U}_2^C(t)|^2 \right) dt \\ & \quad - 2 \int_0^\infty k(t) |\mathcal{U}_1^R(t)|^2 dt. \end{aligned}$$

In virtue of (79) we can control the term  $\int_0^\infty k(t) \mathcal{U}_1^D(t) dt$  (resp.  $\int_0^\infty k(t) \mathcal{U}_2^C(t) dt$ ) by means of  $\int_0^\infty k(t) \mathcal{U}_2^D(t) dt$  (resp.  $\int_0^\infty k(t) \mathcal{U}_1^C(t) dt$ ). Therefore, it is convenient to write the previous formula in the following way

$$\begin{aligned} & \int_0^\infty k(t) |\mathcal{U}_1^C(t) + \mathcal{U}_1^D(t) + \mathcal{U}_1^R(t)|^2 dt + \int_0^\infty k(t) |\mathcal{U}_2^D(t) + \mathcal{U}_2^C(t)|^2 dt \\ & \geq \frac{1}{2} \int_0^\infty k(t) \left( |\mathcal{U}_1^C(t)|^2 - 2 |\mathcal{U}_2^C(t)|^2 \right) dt + \frac{1}{2} \int_0^\infty k(t) \left( |\mathcal{U}_2^D(t)|^2 - 4 |\mathcal{U}_1^D(t)|^2 \right) dt \\ & \quad - 2 \int_0^\infty k(t) |\mathcal{U}_1^R(t)|^2 dt. \quad (83) \end{aligned}$$

We will give a lower bound estimate for  $\int_0^\infty k(t) |\mathcal{U}_1^C(t)|^2 dt$  and  $\int_0^\infty k(t) |\mathcal{U}_2^D(t)|^2 dt$ , and, on the contrary, an upper bound estimate for  $\int_0^\infty k(t) |\mathcal{U}_2^C(t)|^2 dt$ ,  $\int_0^\infty k(t) |\mathcal{U}_1^D(t)|^2 dt$  and  $\int_0^\infty k(t) |\mathcal{U}_1^R(t)|^2 dt$ . So, thanks to (83), we will be able to prove an inverse estimate.

Moreover, if we will assume an additional condition on the coefficients of the series, we will be able to prove an inverse inequality with a better estimate for the control time. Indeed, the additional assumption will allow us to control all terms  $\int_0^\infty k(t) |\mathcal{U}_1^D(t)|^2 dt$ ,  $\int_0^\infty k(t) |\mathcal{U}_2^C(t)|^2 dt$  and  $\int_0^\infty k(t) |\mathcal{U}_1^R(t)|^2 dt$  by means of  $\int_0^\infty k(t) |\mathcal{U}_2^D(t)|^2 dt$ . In this way the estimate of the term  $\int_0^\infty k(t) |\mathcal{U}_1^C(t)|^2 dt$  can be done with the help of an idea used previously in [23]. In fact in this case we will use the following inequality

$$\begin{aligned} & \int_0^\infty k(t) |\mathcal{U}_1^C(t) + \mathcal{U}_1^D(t) + \mathcal{U}_1^R(t)|^2 dt + \int_0^\infty k(t) |\mathcal{U}_2^D(t) + \mathcal{U}_2^C(t)|^2 dt \\ & \geq \frac{1}{2} \int_0^\infty k(t) |\mathcal{U}_1^C(t)|^2 dt + \frac{1}{2} \int_0^\infty k(t) \left( |\mathcal{U}_2^D(t)|^2 - 4 |\mathcal{U}_1^D(t)|^2 - 2 |\mathcal{U}_2^C(t)|^2 - 4 |\mathcal{U}_1^R(t)|^2 \right) dt. \quad (84) \end{aligned}$$

## 5.2 Technical results

In order to avoid repetitions and simplify the proofs of the main theorems, we prefer to single out some lemmas that we will employ in several situations. For this reason, in this subsection we collect some results to be used later.

Let  $T > 0$ . We introduce an auxiliary function defined by

$$k(t) := \begin{cases} \sin \frac{\pi t}{T} & \text{if } t \in [0, T], \\ 0 & \text{otherwise.} \end{cases} \quad (85)$$

In the following lemma we list some useful properties of  $k$ .

**Lemma 5.1** *Set*

$$K(w) := \frac{T\pi}{\pi^2 - T^2 w^2}, \quad w \in \mathbb{C}, \quad (86)$$

*the following properties hold.*

(i) For any  $w \in \mathbb{C}$  one has

$$\overline{K(w)} = K(\overline{w}), \quad |K(w)| = |K(\overline{w})|, \quad (87)$$

$$\int_0^\infty k(t)e^{iwt}dt = (1 + e^{iwT})K(w). \quad (88)$$

(ii) For any  $z_i, w_i \in \mathbb{C}$ ,  $i = 1, 2$ , one has

$$\begin{aligned} & \int_0^\infty k(t)\Re(z_1 e^{iw_1 t})\Re(z_2 e^{iw_2 t})dt \\ &= \frac{1}{2}\Re\left(z_1 z_2(1 + e^{i(w_1 + w_2)T})K(w_1 + w_2) + z_1 \overline{z_2}(1 + e^{i(w_1 - \overline{w_2})T})K(w_1 - \overline{w_2})\right). \end{aligned} \quad (89)$$

(iii) Let  $\overline{\gamma} > 0$  and  $j \in \mathbb{N}$ . Then for  $T > 2\pi/\overline{\gamma}$  and  $w \in \mathbb{C}$ ,  $|w| \geq \overline{\gamma}j$ , one has

$$|K(w)| \leq \frac{4\pi}{T\overline{\gamma}^2(4j^2 - 1)}. \quad (90)$$

*Proof.* (i) The proof is straightforward.

(ii) We note that for any  $z, w \in \mathbb{C}$

$$\int_0^\infty k(t)\Re(ze^{iwt})dt = \Re(z(1 + e^{iwT})K(w)).$$

Therefore, taking into account

$$\Re(z_1 e^{iw_1 t})\Re(z_2 e^{iw_2 t}) = \frac{1}{2}\Re(z_1 z_2 e^{i(w_1 + w_2)t} + z_1 \overline{z_2} e^{i(w_1 - \overline{w_2})t}),$$

it follows (89).

(iii) We observe that

$$|K(w)| = \frac{\pi}{T\left|w^2 - \left(\frac{\pi}{T}\right)^2\right|} = \frac{4\pi}{T\overline{\gamma}^2\left|4\left(\frac{w}{\overline{\gamma}}\right)^2 - \left(\frac{2\pi}{T\overline{\gamma}}\right)^2\right|}.$$

Since  $|w| \geq \overline{\gamma}j$  and  $\frac{2\pi}{T\overline{\gamma}} < 1$ , we have

$$\left|4\left(\frac{w}{\overline{\gamma}}\right)^2 - \left(\frac{2\pi}{T\overline{\gamma}}\right)^2\right| \geq 4\frac{|w|^2}{\overline{\gamma}^2} - \left(\frac{2\pi}{T\overline{\gamma}}\right)^2 \geq 4j^2 - 1,$$

and hence (90) holds true.  $\square$

**Lemma 5.2** *If  $\gamma > 0$  is such that*

$$\liminf_{n \rightarrow \infty} (\Re\sigma_{n+1} - \Re\sigma_n) = \gamma,$$

*then for any  $\varepsilon \in (0, 1)$  there exists  $n_0 \in \mathbb{N}$  such that*

$$|\Re\sigma_n - \Re\sigma_m| \geq \gamma\sqrt{1 - \varepsilon}|n - m|, \quad \forall n, m \geq n_0, \quad (91)$$

$$\Re\sigma_n \geq \gamma\sqrt{1 - \varepsilon}n, \quad \forall n \geq n_0. \quad (92)$$

*Proof.* For  $\varepsilon \in (0, 1)$  there exists  $n_0 \in \mathbb{N}$  such that

$$\Re\sigma_{n+1} - \Re\sigma_n \geq \gamma\sqrt{1 - \varepsilon} \quad \forall n \geq n_0,$$

whence (91) follows. Moreover, in view of

$$\liminf_{n \rightarrow \infty} \frac{\Re\sigma_{n+1}}{n+1} \geq \liminf_{n \rightarrow \infty} (\Re\sigma_{n+1} - \Re\sigma_n), \quad (93)$$

see [2, p. 54], (92) holds true.  $\square$

**Lemma 5.3** (i) For any  $n_0 \in \mathbb{N}$  and  $n \geq n_0$  we have

$$\sum_{\substack{m=n_0 \\ m \neq n}}^{\infty} \frac{1}{4(m-n)^2-1} \leq 1. \quad (94)$$

(ii) Fixed  $a, b \geq 0$  and  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  large enough to satisfy

$$\frac{a}{4n^2-1} + b \sum_{m=n_0}^{\infty} \frac{1}{4m^2-1} \leq \varepsilon \quad \forall n \geq n_0. \quad (95)$$

(iii) Fixed  $a \geq 0$ ,  $\nu > 1/2$  and  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  large enough to satisfy

$$a \sum_{n=n_0}^{\infty} \frac{1}{n^{2\nu}} \leq \varepsilon. \quad (96)$$

*Proof.* (i) We have

$$\begin{aligned} \sum_{\substack{m=n_0 \\ m \neq n}}^{\infty} \frac{1}{4(m-n)^2-1} &= \sum_{m=n_0}^{n-1} \frac{1}{4(n-m)^2-1} + \sum_{m=n+1}^{\infty} \frac{1}{4(m-n)^2-1} \\ &\leq 2 \sum_{j=1}^{\infty} \frac{1}{4j^2-1} = \sum_{j=1}^{\infty} \left( \frac{1}{2j-1} - \frac{1}{2j+1} \right) = 1. \end{aligned}$$

(ii) We observe that for  $n \geq n_0$  we have

$$4n^2-1 \geq 4n^{3/2}n_0^{1/2}-1 \geq n_0^{1/2}(4n^{3/2}-1),$$

and hence

$$\frac{a}{4n^2-1} + b \sum_{m=n_0}^{\infty} \frac{1}{4m^2-1} \leq \frac{1}{n_0^{1/2}} \left( a + b \sum_{m=1}^{\infty} \frac{1}{4m^{3/2}-1} \right).$$

In conclusion, if one takes  $n_0 \in \mathbb{N}$  such that

$$n_0 \geq \frac{1}{\varepsilon^2} \left( a + b \sum_{m=1}^{\infty} \frac{1}{4m^{3/2}-1} \right)^2,$$

then (95) holds true.

(iii) For  $0 < \delta < 2\nu - 1$  we have

$$\sum_{n=n_0}^{\infty} \frac{1}{n^{2\nu}} \leq \frac{1}{n_0^{\delta}} \sum_{n=1}^{\infty} \frac{1}{n^{2\nu-\delta}},$$

whence, for  $n_0 \geq \left( \frac{a}{\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^{2\nu-\delta}} \right)^{1/\delta}$  we have (96).  $\square$

**Lemma 5.4** Suppose that

$$\liminf_{n \rightarrow \infty} (\Re \sigma_{n+1} - \Re \sigma_n) = \gamma > 0.$$

Then for any  $\varepsilon \in (0, 1)$  and  $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$  there exists  $n_0 = n_0(\varepsilon) \in \mathbb{N}$  such that for any  $n \geq n_0$  we have

$$\sum_{\substack{m=n_0 \\ m \neq n}}^{\infty} |K(\sigma_n - \overline{\sigma_m})| + \sum_{m=n_0}^{\infty} |K(\sigma_n + \sigma_m)| \leq \frac{4\pi}{T\gamma^2(1-\varepsilon)} \left( 1 + \sum_{m=n_0}^{\infty} \frac{1}{4m^2-1} \right), \quad (97)$$

*Proof.* As regards the first inequality, we observe that, thanks to (91) and (90), for  $\varepsilon \in (0, 1)$  there exists  $n_0 \in \mathbb{N}$  such that

$$\sum_{\substack{m=n_0 \\ m \neq n}}^{\infty} |K(\sigma_n - \overline{\sigma_m})| \leq \frac{4\pi}{T\gamma^2(1-\varepsilon)} \sum_{\substack{m=n_0 \\ m \neq n}}^{\infty} \frac{1}{4(m-n)^2-1},$$

whence, in view of (94) we get our statement.

Moreover, concerning the second estimate, thanks to (92), we have

$$|\sigma_n + \sigma_m| \geq \Re \sigma_m \geq \gamma \sqrt{1-\varepsilon} m, \quad \forall m \geq n_0.$$

Therefore, using again (90) we obtain the required inequality.  $\square$

The following result is an useful tool in the proof of the Ingham type inverse estimates. For the sake of completeness we prefer to give a detailed proof, although it could be deduced from previous papers, see [10].

**Proposition 5.5** *Given any  $\gamma > 0$  suppose that*

$$\liminf_{n \rightarrow \infty} (\Re \sigma_{n+1} - \Re \sigma_n) = \gamma$$

*and  $\{F_n\}$  is a complex number sequence such that  $\sum_{n=1}^{\infty} |F_n|^2 < +\infty$ .*

*Then for any  $\varepsilon \in (0, 1)$  and  $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$  there exists  $n_0 = n_0(\varepsilon) \in \mathbb{N}$  independent of  $T$  and  $F_n$  such that we have*

$$\begin{aligned} \int_0^{\infty} k(t) \left| \sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t} \right|^2 dt \\ \geq 2\pi T \sum_{n=n_0}^{\infty} \left( \frac{1}{\pi^2 + 4T^2(\Im \sigma_n)^2} - \frac{4}{T^2\gamma^2}(1+\varepsilon) \right) (1 + e^{-2\Im \sigma_n T}) |F_n|^2, \end{aligned} \quad (98)$$

$$\begin{aligned} \int_0^{\infty} k(t) \left| \sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t} \right|^2 dt \\ \leq 2\pi T \sum_{n=n_0}^{\infty} \left( \frac{1}{\pi^2 + 4T^2(\Im \sigma_n)^2} + \frac{4}{T^2\gamma^2}(1+\varepsilon) \right) (1 + e^{-2\Im \sigma_n T}) |F_n|^2. \end{aligned} \quad (99)$$

*Proof.* Let us first observe that

$$\left| \sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t} \right|^2 = 4 \sum_{n,m=n_0}^{\infty} \Re(F_n e^{i\sigma_n t}) \Re(F_m e^{i\sigma_m t}),$$

where  $n_0 \in \mathbb{N}$  will be chosen later. From (89) we have

$$\begin{aligned} \int_0^{\infty} k(t) \left| \sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t} \right|^2 dt \\ = 2 \sum_{n,m=n_0}^{\infty} \Re \left[ F_n \overline{F_m} (1 + e^{i(\sigma_n - \overline{\sigma_m})T}) K(\sigma_n - \overline{\sigma_m}) + F_n F_m (1 + e^{i(\sigma_n + \sigma_m)T}) K(\sigma_n + \sigma_m) \right]. \end{aligned}$$

Since (86) gives  $K(\sigma_n - \overline{\sigma}_n) = \frac{\pi T}{\pi^2 + 4T^2(\Im \sigma_n)^2}$ , it follows that

$$\begin{aligned} & \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty F_n e^{i\sigma_n t} + \overline{F}_n e^{-i\overline{\sigma}_n t} \right|^2 dt - 2\pi T \sum_{n=n_0}^\infty \frac{1 + e^{-2\Im \sigma_n T}}{\pi^2 + 4T^2(\Im \sigma_n)^2} |F_n|^2 \\ &= 2 \sum_{\substack{n, m=n_0 \\ n \neq m}}^\infty \Re[F_n \overline{F}_m (1 + e^{i(\sigma_n - \overline{\sigma}_m)T}) K(\sigma_n - \overline{\sigma}_m)] + 2 \sum_{n, m=n_0}^\infty \Re[F_n F_m (1 + e^{i(\sigma_n + \sigma_m)T}) K(\sigma_n + \sigma_m)]. \end{aligned}$$

Thus

$$\begin{aligned} & \left| \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty F_n e^{i\sigma_n t} + \overline{F}_n e^{-i\overline{\sigma}_n t} \right|^2 dt - 2\pi T \sum_{n=n_0}^\infty \frac{1 + e^{-2\Im \sigma_n T}}{\pi^2 + 4T^2(\Im \sigma_n)^2} |F_n|^2 \right| \\ & \leq 2 \sum_{\substack{n, m=n_0 \\ n \neq m}}^\infty |F_n| |F_m| (1 + e^{-\Im(\sigma_n + \sigma_m)T}) |K(\sigma_n - \overline{\sigma}_m)| \\ & \quad + 2 \sum_{n, m=n_0}^\infty |F_n| |F_m| (1 + e^{-\Im(\sigma_n + \sigma_m)T}) |K(\sigma_n + \sigma_m)|. \quad (100) \end{aligned}$$

By (87) we have

$$|K(\sigma_n - \overline{\sigma}_m)| = |K(\sigma_m - \overline{\sigma}_n)|,$$

hence

$$\begin{aligned} & \sum_{\substack{n, m=n_0 \\ n \neq m}}^\infty |F_n| |F_m| |K(\sigma_n - \overline{\sigma}_m)| \leq \frac{1}{2} \sum_{\substack{n, m=n_0 \\ n \neq m}}^\infty (|F_n|^2 + |F_m|^2) |K(\sigma_n - \overline{\sigma}_m)| \\ &= \frac{1}{2} \sum_{n=n_0}^\infty |F_n|^2 \sum_{\substack{m=n_0 \\ m \neq n}}^\infty |K(\sigma_n - \overline{\sigma}_m)| + \frac{1}{2} \sum_{m=n_0}^\infty |F_m|^2 \sum_{\substack{n=n_0 \\ n \neq m}}^\infty |K(\sigma_m - \overline{\sigma}_n)| \\ &= \sum_{n=n_0}^\infty |F_n|^2 \sum_{\substack{m=n_0 \\ m \neq n}}^\infty |K(\sigma_n - \overline{\sigma}_m)|. \end{aligned}$$

In the same manner we can see that

$$\begin{aligned} & \sum_{\substack{n, m=n_0 \\ n \neq m}}^\infty |F_n| |F_m| e^{-\Im(\sigma_n + \sigma_m)T} |K(\sigma_n - \overline{\sigma}_m)| \leq \sum_{n=n_0}^\infty e^{-2\Im \sigma_n T} |F_n|^2 \sum_{\substack{m=n_0 \\ m \neq n}}^\infty |K(\sigma_n - \overline{\sigma}_m)|, \\ & \sum_{n, m=n_0}^\infty |F_n| |F_m| (1 + e^{-\Im(\sigma_n + \sigma_m)T}) |K(\sigma_n + \sigma_m)| \leq \sum_{n=n_0}^\infty (1 + e^{-2\Im \sigma_n T}) |F_n|^2 \sum_{m=n_0}^\infty |K(\sigma_n + \sigma_m)|. \end{aligned}$$

Substituting these inequalities into (100) yields

$$\begin{aligned} & \left| \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty F_n e^{i\sigma_n t} + \overline{F}_n e^{-i\overline{\sigma}_n t} \right|^2 dt - 2\pi T \sum_{n=n_0}^\infty \frac{1 + e^{-2\Im \sigma_n T}}{\pi^2 + 4T^2(\Im \sigma_n)^2} |F_n|^2 \right| \\ & \leq 2 \sum_{n=n_0}^\infty (1 + e^{-2\Im \sigma_n T}) |F_n|^2 \left( \sum_{\substack{m=n_0 \\ m \neq n}}^\infty |K(\sigma_n - \overline{\sigma}_m)| + \sum_{m=n_0}^\infty |K(\sigma_n + \sigma_m)| \right). \end{aligned}$$

Fix now  $\varepsilon \in (0, 1)$  and  $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$ . As for  $\varepsilon' \in (0, \varepsilon)$  one has  $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon'}}$  too, we can employ Lemma 5.4 with  $\varepsilon$  replaced by  $\varepsilon'$ . Thus taking  $n_0$  as in Lemma 5.4 and applying (97) we obtain

$$\begin{aligned} \left| \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t} \right|^2 dt - 2\pi T \sum_{n=n_0}^\infty \frac{1 + e^{-2\Im\sigma_n T}}{\pi^2 + 4T^2(\Im\sigma_n)^2} |F_n|^2 \right| \\ \leq \frac{8\pi}{T\gamma^2(1-\varepsilon')} \sum_{n=n_0}^\infty (1 + e^{-2\Im\sigma_n T}) |F_n|^2 \left( 1 + \sum_{m=n_0}^\infty \frac{1}{4m^2 - 1} \right). \end{aligned}$$

By Lemma 5.3-(ii) with  $a = 0$  and  $b = 1$  one can pick  $n_0 \in \mathbb{N}$  large enough to satisfy

$$\sum_{m=n_0}^\infty \frac{1}{4m^2 - 1} \leq \varepsilon'.$$

Therefore

$$\begin{aligned} \left| \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t} \right|^2 dt - 2\pi T \sum_{n=n_0}^\infty \frac{1 + e^{-2\Im\sigma_n T}}{\pi^2 + 4T^2(\Im\sigma_n)^2} |F_n|^2 \right| \\ \leq \frac{8\pi}{T\gamma^2} \frac{1 + \varepsilon'}{1 - \varepsilon'} \sum_{n=n_0}^\infty (1 + e^{-2\Im\sigma_n T}) |F_n|^2. \end{aligned}$$

Taking  $\varepsilon' \in (0, \varepsilon)$  such that  $\frac{1+\varepsilon'}{1-\varepsilon'} < 1 + \varepsilon$ , that is  $\varepsilon' < \frac{\varepsilon}{2+\varepsilon}$ , we obtain

$$\begin{aligned} \left| \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t} \right|^2 dt - 2\pi T \sum_{n=n_0}^\infty \frac{1 + e^{-2\Im\sigma_n T}}{\pi^2 + 4T^2(\Im\sigma_n)^2} |F_n|^2 \right| \\ \leq \frac{8\pi}{T\gamma^2} (1 + \varepsilon) \sum_{n=n_0}^\infty (1 + e^{-2\Im\sigma_n T}) |F_n|^2, \end{aligned}$$

which gives (98) and (99).  $\square$

### 5.3 Inverse inequality

Following the outline shown in Section 5.1 we have to estimate all three integrals on the right-hand side of (83). For this reason, for any term to bound we will establish a corresponding lemma.

**Lemma 5.6** *For any  $\varepsilon \in (0, 1)$  and  $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$  there exists  $n_0 = n_0(\varepsilon) \in \mathbb{N}$  independent of  $T$  and  $C_n$  such that we have*

$$\begin{aligned} \int_0^\infty k(t) \left( \left| \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} \right|^2 - 2 \left| \sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 \right) dt \\ \geq 2\pi T \sum_{n=n_0}^\infty \left( \frac{1 - \varepsilon}{\pi^2 + 4T^2(\Im\omega_n)^2} - \frac{4}{T^2\gamma^2} (1 + \varepsilon) \right) (1 + e^{-2\Im\omega_n T}) |C_n|^2. \quad (101) \end{aligned}$$

*Proof.* Fix  $\varepsilon \in (0, 1)$  and  $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$ . Let us apply Proposition 5.5 with  $\sigma_n = \omega_n$ . Indeed, for  $\varepsilon' \in (0, \varepsilon)$  to be chosen later there exists  $n_0$  independent of  $T$  and  $C_n$  such that from (98) with  $F_n = C_n$  and (99) with  $F_n = c_n C_n$  respectively we have

$$\begin{aligned} \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} \right|^2 dt \\ \geq 2\pi T \sum_{n=n_0}^\infty \left( \frac{1}{\pi^2 + 4T^2(\Im\omega_n)^2} - \frac{4}{T^2\gamma^2} (1 + \varepsilon') \right) (1 + e^{-2\Im\omega_n T}) |C_n|^2, \quad (102) \end{aligned}$$

$$\begin{aligned} \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 dt \\ \leq 2\pi T \sum_{n=n_0}^\infty \left( \frac{1}{\pi^2 + 4T^2(\Im \omega_n)^2} + \frac{4}{T^2 \gamma^2} (1 + \varepsilon') \right) (1 + e^{-2\Im \omega_n T}) |c_n C_n|^2. \end{aligned} \quad (103)$$

Combining these inequalities gives

$$\begin{aligned} \int_0^\infty k(t) \left( \left| \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} \right|^2 - 2 \left| \sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 \right) dt \\ \geq 2\pi T \sum_{n=n_0}^\infty \left( \frac{1 - 2|c_n|^2}{\pi^2 + 4T^2(\Im \omega_n)^2} - \frac{4}{T^2 \gamma^2} (1 + \varepsilon') (1 + 2|c_n|^2) \right) (1 + e^{-2\Im \omega_n T}) |C_n|^2. \end{aligned}$$

We will choose  $\varepsilon'$  in a suitable way to obtain our statement. Thanks to (79) for  $n_0$  large enough we have  $2|c_n|^2 \leq \varepsilon'$  for  $n \geq n_0$ . Hence

$$(1 + \varepsilon')(1 + 2|c_n|^2) \leq (1 + \varepsilon')^2 \leq 1 + 3\varepsilon' \quad \forall n \geq n_0.$$

Taking  $\varepsilon' < \varepsilon/3$  yields

$$(1 + \varepsilon')(1 + 2|c_n|^2) \leq 1 + \varepsilon \quad \forall n \geq n_0.$$

Moreover, since  $2|c_n|^2 \leq \varepsilon$  we get (101) and the proof is complete.  $\square$

To estimate the second integral on the right-hand side of (83) we state the following result, that may be proved in much the same way as the previous lemma by means of Proposition 5.5 with  $\sigma_n = \zeta_n$  and (79). For this reason we omit the proof.

**Lemma 5.7** *For any  $\varepsilon \in (0, 1)$  and  $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$  there exists  $n_0 = n_0(\varepsilon) \in \mathbb{N}$  independent of  $T$  and  $D_n$  such that we have*

$$\begin{aligned} \int_0^\infty k(t) \left( \left| \sum_{n=n_0}^\infty d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} \right|^2 - 4 \left| \sum_{n=n_0}^\infty D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 \right) dt \\ \geq 2\pi T \sum_{n=n_0}^\infty \left( \frac{1 - \varepsilon}{\pi^2 + 4T^2(\Im \zeta_n)^2} - \frac{4}{T^2 \gamma^2} (1 + \varepsilon) \right) (1 + e^{-2\Im \zeta_n T}) |d_n D_n|^2. \end{aligned} \quad (104)$$

Finally, we will give an estimate for the last integral on the right-hand side of (83).

**Lemma 5.8** *For any  $\varepsilon \in (0, 1)$  and  $T > 0$  there exists  $n_0 = n_0(\varepsilon) \in \mathbb{N}$  independent of  $T$  and  $R_n$  such that we have*

$$\int_0^\infty k(t) \left| \sum_{n=n_0}^\infty R_n e^{r_n t} \right|^2 dt \leq \varepsilon \pi T \sum_{n=n_0}^\infty \frac{|C_n|^2 + |d_n D_n|^2}{\pi^2 + T^2 r_n^2}. \quad (105)$$

*Proof.* Our proof starts with the observation that (88) leads to

$$\begin{aligned} \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty R_n e^{r_n t} \right|^2 dt &= \sum_{n,m=n_0}^\infty R_n R_m \int_0^\infty k(t) e^{(r_n + r_m)t} dt \\ &= \sum_{n,m=n_0}^\infty R_n R_m (1 + e^{(r_n + r_m)T}) K(ir_n + ir_m), \end{aligned}$$

where  $n_0 \in \mathbb{N}$  has to be chosen later. By the definition (86) of  $K$  we have

$$K(ir_n + ir_m) = \frac{T\pi}{\pi^2 + T^2(r_n + r_m)^2}.$$



Let us apply  $r_n \leq 0$  for  $n \geq n'$  to obtain

$$1 + e^{(r_n + r_m)T} \leq 2.$$

Consequently, taking  $n_0 \geq n'$  we get

$$\int_0^\infty k(t) \left| \sum_{n=n_0}^\infty R_n e^{r_n t} \right|^2 dt \leq 2\pi T \sum_{n,m=n_0}^\infty \frac{|R_n| |R_m|}{\pi^2 + T^2(r_n + r_m)^2}.$$

From (80) we see that

$$\begin{aligned} \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty R_n e^{r_n t} \right|^2 dt \\ \leq 2\pi T \mu^2 \sum_{n,m=n_0}^\infty \frac{(|C_n|^2 + |d_n D_n|^2)^{1/2}}{m^\nu} \frac{(|C_m|^2 + |d_m D_m|^2)^{1/2}}{n^\nu} \frac{1}{\pi^2 + T^2(r_n + r_m)^2}. \end{aligned}$$

Using again (78) yields

$$\begin{aligned} \sum_{n,m=n_0}^\infty \frac{(|C_n|^2 + |d_n D_n|^2)^{1/2}}{m^\nu} \frac{(|C_m|^2 + |d_m D_m|^2)^{1/2}}{n^\nu} \frac{1}{\pi^2 + T^2(r_n + r_m)^2} \\ \leq \frac{1}{2} \sum_{m=n_0}^\infty \frac{1}{m^{2\nu}} \sum_{n=n_0}^\infty \frac{|C_n|^2 + |d_n D_n|^2}{\pi^2 + T^2 r_n^2} + \frac{1}{2} \sum_{n=n_0}^\infty \frac{1}{n^{2\nu}} \sum_{m=n_0}^\infty \frac{|C_m|^2 + |d_m D_m|^2}{\pi^2 + T^2 r_m^2} \\ = \sum_{n=n_0}^\infty \frac{1}{n^{2\nu}} \sum_{n=n_0}^\infty \frac{|C_n|^2 + |d_n D_n|^2}{\pi^2 + T^2 r_n^2}. \end{aligned}$$

Combining these inequalities we deduce that

$$\int_0^\infty k(t) \left| \sum_{n=n_0}^\infty R_n e^{r_n t} \right|^2 dt \leq 2\pi T \mu^2 \sum_{n=n_0}^\infty \frac{1}{n^{2\nu}} \sum_{n=n_0}^\infty \frac{|C_n|^2 + |d_n D_n|^2}{\pi^2 + T^2 r_n^2}.$$

Applying Lemma 5.3-(iii) we conclude that (105) is proved.  $\square$

We will establish the main result to obtain the inverse inequality. To simplify our notations, in the following we will use the symbols

$$\begin{aligned} u_1^{n_0}(t) &:= \sum_{n=n_0}^\infty \left( C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right), \\ u_2^{n_0}(t) &:= \sum_{n=n_0}^\infty \left( d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} + c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right), \end{aligned} \tag{106}$$

**Theorem 5.9** Assume  $\gamma > 4\alpha$ . Then for any  $\varepsilon \in (0, \frac{\gamma^2 - 16\alpha^2}{\gamma^2 + 16\alpha^2})$  and  $T > \frac{2\pi}{\sqrt{\gamma^2(1-\varepsilon) - 16\alpha^2(1+\varepsilon)}}$  there exist  $n_0 = n_0(\varepsilon) \in \mathbb{N}$ , independent of  $T$  and all coefficients of the series, and a constant  $c(T, \varepsilon) > 0$  such that

$$\begin{aligned} \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 dt \\ + \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} + c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 dt \\ \geq c(T, \varepsilon) \sum_{n=n_0}^\infty (1 + e^{-2\Im \omega_n T}) (|C_n|^2 + |d_n D_n|^2). \end{aligned} \tag{107}$$

*Proof.* Fix  $\varepsilon \in (0, 1)$ , in view of (106) our goal is to evaluate the following sum

$$\int_0^\infty k(t) (|u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2) dt, \quad (108)$$

where the index  $n_0 \in \mathbb{N}$  depending on  $\varepsilon$  will be chosen suitably. To this end, we bear in mind the comments given in Section 5.1. Indeed, we observe that

$$\begin{aligned} & \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 dt \\ & \geq \frac{1}{2} \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} \right|^2 dt - 2 \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 dt \\ & \quad - 2 \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty R_n e^{r_n t} \right|^2 dt \end{aligned}$$

and

$$\begin{aligned} & \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} + c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 dt \\ & \geq \frac{1}{2} \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} \right|^2 dt - \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 dt. \end{aligned}$$

Combining these inequalities we obtain

$$\begin{aligned} & \int_0^\infty k(t) (|u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2) dt \\ & \geq \frac{1}{2} \int_0^\infty k(t) \left( \left| \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} \right|^2 - 2 \left| \sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 \right) dt \\ & \quad + \frac{1}{2} \int_0^\infty k(t) \left( \left| \sum_{n=n_0}^\infty d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} \right|^2 - 4 \left| \sum_{n=n_0}^\infty D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 \right) dt \\ & \quad - 2 \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty R_n e^{r_n t} \right|^2 dt. \end{aligned}$$

We now take  $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$  to estimate the first two integrals on the right-hand side. We introduce  $\varepsilon' \in (0, \varepsilon)$  to choose suitably later. We also have  $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon'}}$ , so we can use (101) and (104) respectively to obtain

$$\begin{aligned} & \int_0^\infty k(t) \left( \left| \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} \right|^2 - 2 \left| \sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 \right) dt \\ & \geq 2\pi T \sum_{n=n_0}^\infty \left( \frac{1-\varepsilon'}{\pi^2 + 4T^2(\Im \omega_n)^2} - \frac{4}{T^2 \gamma^2} (1+\varepsilon') \right) (1 + e^{-2\Im \omega_n T}) |C_n|^2, \\ & \int_0^\infty k(t) \left( \left| \sum_{n=n_0}^\infty d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} \right|^2 - 4 \left| \sum_{n=n_0}^\infty D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 \right) dt \\ & \geq 2\pi T \sum_{n=n_0}^\infty \left( \frac{1-\varepsilon'}{\pi^2 + 4T^2(\Im \zeta_n)^2} - \frac{4}{T^2 \gamma^2} (1+\varepsilon') \right) (1 + e^{-2\Im \zeta_n T}) |d_n D_n|^2. \end{aligned}$$

By (78) we get  $|\Im \zeta_n| \leq \Im \omega_n$  for  $n \geq n_0$  with  $n_0$  sufficiently large. Hence

$$\frac{e^{-2\Im \zeta_n T}}{\pi^2 + 4T^2(\Im \zeta_n)^2} \geq \frac{e^{-2\Im \omega_n T}}{\pi^2 + 4T^2(\Im \omega_n)^2} \quad \forall n \geq n_0.$$

Therefore

$$\begin{aligned} & \int_0^\infty k(t) (|u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2) dt \\ & \geq \pi T \sum_{n=n_0}^\infty \left( \frac{1 - \varepsilon'}{\pi^2 + 4T^2(\Im \omega_n)^2} - \frac{4}{T^2 \gamma^2} (1 + \varepsilon') \right) (1 + e^{-2\Im \omega_n T}) (|C_n|^2 + |d_n D_n|^2) \\ & \quad - 2 \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty R_n e^{r_n t} \right|^2 dt. \end{aligned}$$

Applying (105) we obtain

$$\begin{aligned} & \int_0^\infty k(t) (|u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2) dt \\ & \geq \pi T \sum_{n=n_0}^\infty \left( \frac{1 - \varepsilon'}{\pi^2 + 4T^2(\Im \omega_n)^2} - \frac{\varepsilon'}{\pi^2 + T^2 r_n^2} - \frac{4}{T^2 \gamma^2} (1 + \varepsilon') \right) (1 + e^{-2\Im \omega_n T}) (|C_n|^2 + |d_n D_n|^2). \quad (109) \end{aligned}$$

Now, we will choose  $\varepsilon' \in (0, \varepsilon)$  such that for  $n \geq n_0$

$$\frac{1 - \varepsilon'}{\pi^2 + 4T^2(\Im \omega_n)^2} - \frac{\varepsilon'}{\pi^2 + T^2 r_n^2} \geq \frac{1 - \varepsilon}{\pi^2 + 4T^2(\Im \omega_n)^2}, \quad (110)$$

that is

$$\begin{aligned} & \frac{\varepsilon - \varepsilon'}{\pi^2 + 4T^2(\Im \omega_n)^2} - \frac{\varepsilon'}{\pi^2 + T^2 r_n^2} \geq 0, \\ & \pi^2(\varepsilon - 2\varepsilon') + T^2[(\varepsilon - \varepsilon')r_n^2 - 4\varepsilon'(\Im \omega_n)^2] \geq 0. \end{aligned}$$

To this end, we need to have that

$$\varepsilon - 2\varepsilon' \geq 0, \quad (\varepsilon - \varepsilon')r_n^2 - 4\varepsilon'(\Im \omega_n)^2 \geq 0. \quad (111)$$

By (78) for  $n_0$  sufficiently large we have

$$r_n^2 \geq \frac{\chi^2}{2}, \quad (\Im \omega_n)^2 \leq \frac{3}{2}\alpha^2.$$

Hence

$$(\varepsilon - \varepsilon')r_n^2 - 4\varepsilon'(\Im \omega_n)^2 \geq (\varepsilon - \varepsilon')\frac{\chi^2}{2} - 6\varepsilon'\alpha^2.$$

Therefore taking

$$\varepsilon' \leq \min \left\{ \frac{1}{2}, \frac{\chi^2}{\chi^2 + 12\alpha^2} \right\} \varepsilon,$$

we deduce (111), and consequently (110). So, from (109) we have

$$\begin{aligned} & \int_0^\infty k(t) (|u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2) dt \\ & \geq \pi T \sum_{n=n_0}^\infty \left( \frac{1 - \varepsilon}{\pi^2 + 4T^2(\Im \omega_n)^2} - \frac{4}{T^2 \gamma^2} (1 + \varepsilon) \right) (1 + e^{-2\Im \omega_n T}) (|C_n|^2 + |d_n D_n|^2). \end{aligned}$$

Since the previous inequality holds for any  $\varepsilon \in (0, 1)$ , in particular it can be written for  $\varepsilon' < \frac{\varepsilon}{2-\varepsilon}$ , because this implies  $\frac{1+\varepsilon'}{1-\varepsilon'} < \frac{1}{1-\varepsilon}$ , and hence

$$\frac{1-\varepsilon'}{\pi^2 + 4T^2(\Im\omega_n)^2} - \frac{4}{T^2\gamma^2}(1+\varepsilon') \geq (1-\varepsilon') \left( \frac{1}{\pi^2 + 4T^2(\Im\omega_n)^2} - \frac{4}{T^2\gamma^2(1-\varepsilon)} \right).$$

Therefore, taking also into account that  $(\Im\omega_n)^2 < \alpha^2(1+\varepsilon)$ ,  $n \geq n_0$ , for  $n_0$  large enough, we can write

$$\begin{aligned} & \int_0^\infty k(t) (|u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2) dt \\ & \geq \pi T(1-\varepsilon') \left( \frac{1}{\pi^2 + 4T^2\alpha^2(1+\varepsilon)} - \frac{4}{T^2\gamma^2(1-\varepsilon)} \right) \sum_{n=n_0}^\infty (1 + e^{-2\Im\omega_n T}) (|C_n|^2 + |d_n D_n|^2). \end{aligned} \quad (112)$$

The constant

$$\frac{1}{\pi^2 + 4T^2\alpha^2(1+\varepsilon)} - \frac{4}{T^2\gamma^2(1-\varepsilon)}$$

is positive if

$$T^2[\gamma^2(1-\varepsilon) - 16\alpha^2(1+\varepsilon)] > 4\pi^2. \quad (113)$$

Since  $\gamma > 4\alpha$  we have  $\gamma^2(1-\varepsilon) - 16\alpha^2(1+\varepsilon) > 0$  if  $\varepsilon < \frac{\gamma^2 - 16\alpha^2}{\gamma^2 + 16\alpha^2}$ . If we assume the more restrictive condition  $T > \frac{2\pi}{\sqrt{\gamma^2(1-\varepsilon) - 16\alpha^2(1+\varepsilon)}}$  with respect to that  $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$ , then (113) holds true. Finally, from (112) and the definition (108) of  $\mathcal{I}_{n_0}$  we obtain (107).  $\square$

We now observe that we can obtain a better estimate of the control time  $T$  under an additional condition on the coefficients of the series. Assuming  $|C_n| \leq M|d_n D_n|$ , we can follow the procedure sketched out at the end of Section 5.1 by using estimate (84). In particular, to evaluate the term  $\int_0^\infty k(t) |\mathcal{U}_1^C(t)|^2 dt$  we will employ the same trick used in [23], giving first an estimate for  $\int_0^\infty e^{2\alpha t} k(t) |\mathcal{U}_1^C(t)|^2 dt$  where  $\alpha = \lim_{n \rightarrow \infty} \Im\omega_n$  and then multiplying by  $e^{-2\alpha T}$  we will obtain the requested inequality.

**Theorem 5.10** *Assume*

$$|C_n| \leq M|d_n D_n| \quad \forall n \in \mathbb{N}. \quad (114)$$

*Then, for any  $\varepsilon \in (0, 1)$  and  $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$  there exist  $n_0 = n_0(\varepsilon) \in \mathbb{N}$ , independent of  $T$  and all coefficients of the series, and a constant  $c(T, \varepsilon) > 0$  such that*

$$\begin{aligned} & \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 dt \\ & + \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} + c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 dt \\ & \geq c(T, \varepsilon) \sum_{n=n_0}^\infty (|C_n|^2 + |d_n D_n|^2). \end{aligned} \quad (115)$$

*Proof.* If  $\alpha = \lim_{n \rightarrow \infty} \Im\omega_n$ , see (78), since

$$\int_0^\infty e^{2\alpha t} k(t) \left| \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} \right|^2 dt = \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty C_n e^{i(\omega_n - i\alpha)t} + \overline{C_n} e^{-i(\overline{\omega_n} - i\alpha)t} \right|^2 dt,$$

thanks to (98) we have

$$\begin{aligned} \int_0^\infty e^{2\alpha t} k(t) \left| \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega}_n t} \right|^2 dt \\ \geq 2\pi T \sum_{n=n_0}^\infty \left( \frac{1}{\pi^2 + 4T^2(\Im \omega_n - \alpha)^2} - \frac{4}{T^2 \gamma^2} (1 + \varepsilon') \right) (1 + e^{-2(\Im \omega_n - \alpha)T}) |C_n|^2, \end{aligned}$$

where  $\varepsilon' \in (0, \varepsilon)$  will be chosen later. Therefore, multiplying by  $e^{-2\alpha T}$  and taking into account the definition (85) of the function  $k$ , we get

$$\begin{aligned} \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega}_n t} \right|^2 dt \\ \geq 2\pi T e^{-2\alpha T} \sum_{n=n_0}^\infty \left( \frac{1}{\pi^2 + 4T^2(\Im \omega_n - \alpha)^2} - \frac{4}{T^2 \gamma^2} (1 + \varepsilon') \right) (1 + e^{-2(\Im \omega_n - \alpha)T}) |C_n|^2. \end{aligned}$$

Now, we can take  $4(\Im \omega_n - \alpha)^2 < \gamma^2 \varepsilon / 8$  for  $n \geq n_0$  and  $1 + \varepsilon' < \frac{1}{1 - \varepsilon/2}$  for  $\varepsilon' < \frac{\varepsilon}{2 - \varepsilon}$ , to have

$$\begin{aligned} \frac{1}{\pi^2 + 4T^2(\Im \omega_n - \alpha)^2} - \frac{4}{T^2 \gamma^2} (1 + \varepsilon') \\ > \frac{1}{\pi^2 + T^2 \gamma^2 \varepsilon / 8} - \frac{4}{T^2 \gamma^2 (1 - \varepsilon/2)} = \frac{T^2 \gamma^2 (1 - \varepsilon) - 4\pi^2}{(\pi^2 + T^2 \gamma^2 \varepsilon / 8) T^2 \gamma^2 (1 - \varepsilon/2)} \end{aligned}$$

and  $T^2 \gamma^2 (1 - \varepsilon) - 4\pi^2 > 0$  for  $T > \frac{2\pi}{\gamma \sqrt{1 - \varepsilon}}$ . So, we get

$$\begin{aligned} \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega}_n t} \right|^2 dt \\ \geq 2\pi T e^{-2\alpha T} \frac{T^2 \gamma^2 (1 - \varepsilon) - 4\pi^2}{(\pi^2 + T^2 \gamma^2 \varepsilon / 8) T^2 \gamma^2 (1 - \varepsilon/2)} \sum_{n=n_0}^\infty (1 + e^{-2(\Im \omega_n - \alpha)T}) |C_n|^2. \quad (116) \end{aligned}$$

On the other hand, from (99) it follows

$$\begin{aligned} \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega}_n t} \right|^2 dt \\ \leq 2\pi T \sum_{n=n_0}^\infty \left( \frac{1}{\pi^2 + 4T^2(\Im \omega_n)^2} + \frac{4}{T^2 \gamma^2} (1 + \varepsilon') \right) (1 + e^{-2\Im \omega_n T}) |c_n C_n|^2 \\ \leq 2\pi T \sum_{n=n_0}^\infty M |c_n|^2 \left( \frac{1}{\pi^2 + 4T^2(\Im \zeta_n)^2} + \frac{4}{T^2 \gamma^2} (1 + \varepsilon') \right) (1 + e^{-2\Im \zeta_n T}) |d_n D_n|^2, \end{aligned}$$

thanks also to  $\Im \omega_n \geq |\Im \zeta_n|$  and  $|C_n| \leq M |d_n D_n|$ . Moreover, again by (99) and the previous inequality we have

$$\begin{aligned} \int_0^\infty k(t) \left( 2 \left| \sum_{n=n_0}^\infty D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta}_n t} \right|^2 + \left| \sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega}_n t} \right|^2 \right) dt \\ \leq 2\pi T \sum_{n=n_0}^\infty \left( \frac{2}{|d_n|^2} + M |c_n|^2 \right) \left( \frac{1}{\pi^2 + 4T^2(\Im \zeta_n)^2} + \frac{4}{T^2 \gamma^2} (1 + \varepsilon') \right) (1 + e^{-2\Im \zeta_n T}) |d_n D_n|^2. \end{aligned}$$

Choosing  $n_0$  sufficiently large such that  $\frac{2}{|d_n|^2} + M|c_n|^2 \leq \varepsilon'$  for any  $n \geq n_0$ , from the above estimate we deduce

$$\begin{aligned} \int_0^\infty k(t) \left( 2 \left| \sum_{n=n_0}^\infty D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 + \left| \sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 \right) dt \\ \leq 2\pi T \varepsilon' \sum_{n=n_0}^\infty \left( \frac{1}{\pi^2 + 4T^2(\Im \zeta_n)^2} + \frac{4}{T^2 \gamma^2} (1 + \varepsilon') \right) (1 + e^{-2\Im \zeta_n T}) |d_n D_n|^2. \end{aligned} \quad (117)$$

In addition, from (105), using again  $|C_n| \leq M|d_n D_n|$  and (78) we get

$$\int_0^\infty k(t) \left| \sum_{n=n_0}^\infty R_n e^{r_n t} \right|^2 dt \leq \pi T \varepsilon' \sum_{n=n_0}^\infty \frac{|d_n D_n|^2}{\pi^2 + T^2 r_n^2} \leq \pi T \varepsilon' \sum_{n=n_0}^\infty \frac{|d_n D_n|^2}{\pi^2 + 4T^2(\Im \zeta_n)^2}. \quad (118)$$

Combining (117) and (118) (with  $\varepsilon'$  replaced by  $\varepsilon'/2$ ) we obtain

$$\begin{aligned} \int_0^\infty k(t) \left( 2 \left| \sum_{n=n_0}^\infty D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 + \left| \sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 + 2 \left| \sum_{n=n_0}^\infty R_n e^{r_n t} \right|^2 \right) dt \\ \leq 2\pi T \varepsilon' \sum_{n=n_0}^\infty \left( \frac{1}{\pi^2 + 4T^2(\Im \zeta_n)^2} + \frac{4}{T^2 \gamma^2} (1 + \varepsilon') \right) (1 + e^{-2\Im \zeta_n T}) |d_n D_n|^2. \end{aligned} \quad (119)$$

In virtue of (98) we get

$$\begin{aligned} \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} \right|^2 dt \\ \geq 2\pi T \sum_{n=n_0}^\infty \left( \frac{1}{\pi^2 + 4T^2(\Im \zeta_n)^2} - \frac{4}{T^2 \gamma^2} (1 + \varepsilon') \right) (1 + e^{-2\Im \zeta_n T}) |d_n D_n|^2. \end{aligned}$$

From the above formula and (119), taking  $\varepsilon' \leq \varepsilon/3$  but writing again  $\varepsilon'$  instead of  $\varepsilon$ , we have

$$\begin{aligned} \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} \right|^2 dt \\ - 2 \int_0^\infty k(t) \left( 2 \left| \sum_{n=n_0}^\infty D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 + \left| \sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 + 2 \left| \sum_{n=n_0}^\infty R_n e^{r_n t} \right|^2 \right) dt \\ \geq 2\pi T \sum_{n=n_0}^\infty \left( \frac{1 - \varepsilon'}{\pi^2 + 4T^2(\Im \zeta_n)^2} - \frac{4}{T^2 \gamma^2} (1 + \varepsilon') \right) (1 + e^{-2\Im \zeta_n T}) |d_n D_n|^2. \end{aligned}$$

Taking  $4(\Im \zeta_n)^2 < \gamma^2 \varepsilon/8$  for  $n \geq n_0$  and  $\frac{1+\varepsilon'}{1-\varepsilon'} < \frac{1}{1-\varepsilon/2}$  for  $\varepsilon' < \frac{\varepsilon}{4-\varepsilon}$  yields

$$\begin{aligned} \frac{1 - \varepsilon'}{\pi^2 + 4T^2(\Im \zeta_n)^2} - \frac{4}{T^2 \gamma^2} (1 + \varepsilon') &= (1 - \varepsilon') \left( \frac{1}{\pi^2 + 4T^2(\Im \zeta_n)^2} - \frac{4(1 + \varepsilon')}{T^2 \gamma^2 (1 - \varepsilon')} \right) \\ &\geq (1 - \varepsilon') \left( \frac{1}{\pi^2 + T^2 \gamma^2 \varepsilon/8} - \frac{4}{T^2 \gamma^2 (1 - \varepsilon/2)} \right) = (1 - \varepsilon') \left( \frac{T^2 \gamma^2 (1 - \varepsilon) - 4\pi^2}{(\pi^2 + T^2 \gamma^2 \varepsilon/8) T^2 \gamma^2 (1 - \varepsilon/2)} \right). \end{aligned}$$

Therefore, for  $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$  we obtain

$$\begin{aligned} & \int_0^\infty k(t) \left| \sum_{n=n_0}^\infty d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} \right|^2 dt \\ & - 2 \int_0^\infty k(t) \left( 2 \left| \sum_{n=n_0}^\infty D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 + \left| \sum_{n=n_0}^\infty c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 + 2 \left| \sum_{n=n_0}^\infty R_n e^{r_n t} \right|^2 \right) dt \\ & \geq 2\pi T(1-\varepsilon) \left( \frac{T^2\gamma^2(1-\varepsilon) - 4\pi^2}{(\pi^2 + T^2\gamma^2\varepsilon/8)T^2\gamma^2(1-\varepsilon/2)} \right) \sum_{n=n_0}^\infty (1 + e^{-2\Im\zeta_n T}) |d_n D_n|^2. \end{aligned}$$

In conclusion, for any  $T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$ , combining the previous estimate with (116) gives

$$\begin{aligned} & \int_0^\infty k(t) (|u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2) dt \\ & \geq \pi T \min\{e^{-2\alpha T}, (1-\varepsilon)\} \left( \frac{T^2\gamma^2(1-\varepsilon) - 4\pi^2}{(\pi^2 + T^2\gamma^2\varepsilon/8)T^2\gamma^2(1-\varepsilon/2)} \right) \sum_{n=n_0}^\infty (|C_n|^2 + |d_n D_n|^2), \end{aligned}$$

that is (115).  $\square$

## 5.4 Direct inequality

As for the inverse inequality, to prove direct estimates we need to introduce an auxiliary function. Let  $T > 0$  and define

$$k^*(t) := \begin{cases} \cos \frac{\pi t}{2T} & \text{if } |t| \leq T, \\ 0 & \text{if } |t| > T. \end{cases} \quad (120)$$

For the sake of completeness, we list some standard properties of  $k^*$  in the following lemma.

**Lemma 5.11** *Set*

$$K^*(u) := \frac{4T\pi}{\pi^2 - 4T^2u^2}, \quad u \in \mathbb{C}, \quad (121)$$

*the following properties hold for any  $u \in \mathbb{C}$*

$$\int_{-\infty}^\infty k^*(t) e^{iut} dt = \cos(uT) K^*(u), \quad (122)$$

$$\overline{K^*(u)} = K^*(\overline{u}), \quad |K^*(u)| = |K^*(\overline{u})|. \quad (123)$$

*Set  $K_T(u) = \frac{T\pi}{\pi^2 - T^2u^2}$  we have*

$$K^*(u) = 2K_{2T}(u). \quad (124)$$

*Moreover for any  $z_i, w_i \in \mathbb{C}$ ,  $i = 1, 2$ , one has*

$$\begin{aligned} & \int_{-\infty}^\infty k^*(t) \Re(z_1 e^{iw_1 t}) \Re(z_2 e^{iw_2 t}) dt \\ & = \frac{1}{2} \Re \left( z_1 z_2 \cos((w_1 + w_2)T) K(w_1 + w_2) + z_1 \overline{z_2} \cos((w_1 - \overline{w_2})T) K(w_1 - \overline{w_2}) \right). \end{aligned} \quad (125)$$

From now on we will denote with  $c(T)$  a positive constant depending on  $T$ .

**Proposition 5.12** *Let  $\gamma > 0$ . Suppose that  $\{\sigma_n\}$  is a complex number sequence satisfying*

$$\liminf_{n \rightarrow \infty} (\Re \sigma_{n+1} - \Re \sigma_n) = \gamma, \quad \{\Im \sigma_n\} \text{ bounded.}$$

*Then for any complex number sequence  $\{F_n\}$  with  $\sum_{n=1}^{\infty} |F_n|^2 < +\infty$ ,  $\varepsilon \in (0, 1)$  and  $T > \frac{\pi}{\gamma\sqrt{1-\varepsilon}}$  there exist  $c(T) > 0$  and  $n_0 = n_0(\varepsilon) \in \mathbb{N}$  independent of  $T$  and  $F_n$  such that*

$$\int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t} \right|^2 dt \leq c(T) \sum_{n=n_0}^{\infty} |F_n|^2. \quad (126)$$

*Proof.* Let us first observe that

$$\left| \sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t} \right|^2 = 4 \sum_{n,m=n_0}^{\infty} \Re(F_n e^{i\sigma_n t}) \Re(F_m e^{i\sigma_m t}),$$

where the index  $n_0 \in \mathbb{N}$  depending on  $\varepsilon$  will be chosen later. From (125) we have

$$\begin{aligned} & \int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t} \right|^2 dt \\ &= 2 \sum_{n,m=n_0}^{\infty} \Re \left[ F_n \overline{F_m} \cos((\sigma_n - \overline{\sigma_m})T) K^*(\sigma_n - \overline{\sigma_m}) + F_n F_m \cos((\sigma_n + \sigma_m)T) K^*(\sigma_n + \sigma_m) \right]. \end{aligned}$$

Applying the elementary estimates  $\Re z \leq |z|$  and  $|\cos z| \leq \cosh(\Im z)$ ,  $z \in \mathbb{C}$ , we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t} \right|^2 dt \\ & \leq 2 \sum_{n,m=n_0}^{\infty} |F_n| |F_m| \cosh(\Im(\sigma_n + \sigma_m)T) [ |K^*(\sigma_n - \overline{\sigma_m})| + |K^*(\sigma_n + \sigma_m)| ]. \end{aligned}$$

Since the sequence  $\{\Im \sigma_n\}$  is bounded we have

$$\cosh(\Im(\sigma_n + \sigma_m)T) \leq e^{2T \sup |\Im \sigma_n|} \quad \forall n, m \in \mathbb{N}.$$

Hence

$$\begin{aligned} & \int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t} \right|^2 dt \\ & \leq 2e^{2T \sup |\Im \sigma_n|} \sum_{n,m=n_0}^{\infty} |F_n| |F_m| [ |K^*(\sigma_n - \overline{\sigma_m})| + |K^*(\sigma_n + \sigma_m)| ]. \end{aligned}$$

Thanks to (123) we get  $|K^*(\sigma_n - \overline{\sigma_m})| = |K^*(\sigma_m - \overline{\sigma_n})|$ . Therefore

$$\begin{aligned} & \int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma_n} t} \right|^2 dt \\ & \leq 2e^{2T \sup |\Im \sigma_n|} \sum_{n=n_0}^{\infty} |F_n|^2 \sum_{m=n_0}^{\infty} [ |K^*(\sigma_n - \overline{\sigma_m})| + |K^*(\sigma_n + \sigma_m)| ]. \end{aligned}$$



Since (121) gives

$$K^*(\sigma_n - \overline{\sigma}_n) = \frac{4\pi T}{\pi^2 + 16T^2(\Im \sigma_n)^2} \leq \frac{4T}{\pi},$$

it follows that

$$\begin{aligned} \int_{-\infty}^{\infty} k(t) \left| \sum_{n=n_0}^{\infty} F_n e^{i\sigma_n t} + \overline{F_n} e^{-i\overline{\sigma}_n t} \right|^2 dt &\leq \frac{8}{\pi} e^{2T \sup |\Im \sigma_n|} T \sum_{n=n_0}^{\infty} |F_n|^2 \\ &+ 2e^{2T \sup |\Im \sigma_n|} \sum_{n=n_0}^{\infty} |F_n|^2 \left[ \sum_{\substack{m=n_0 \\ m \neq n}}^{\infty} |K^*(\sigma_n - \overline{\sigma}_m)| + \sum_{m=n_0}^{\infty} K^*(\sigma_n + \sigma_m) \right]. \end{aligned} \quad (127)$$

Note that by (124) we can apply Lemma 5.4: for any  $\varepsilon \in (0, 1)$  and  $2T > \frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$  there exists  $n_0 \in \mathbb{N}$  such that

$$\sum_{\substack{m=n_0 \\ m \neq n}}^{\infty} |K^*(\sigma_n - \overline{\sigma}_m)| + \sum_{m=n_0}^{\infty} K^*(\sigma_n + \sigma_m) \leq \frac{2\pi}{T\gamma^2(1-\varepsilon)} \left( 1 + \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \right).$$

Substituting the previous estimate into (127) gives (126).  $\square$

**Proposition 5.13** *For any  $n_0 \in \mathbb{N}$ ,  $n_0 \geq n'$ , and  $T > 0$  there exists  $c(T) > 0$  such that*

$$\int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} R_n e^{r_n t} \right|^2 dt \leq c(T) \sum_{n=n_0}^{\infty} (|C_n|^2 + |d_n D_n|^2). \quad (128)$$

*Proof.* Fixed  $n_0 \in \mathbb{N}$ ,  $n_0 \geq n'$ , we observe that (122) leads to

$$\begin{aligned} \int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} R_n e^{r_n t} \right|^2 dt &= \sum_{n,m=n_0}^{\infty} R_n R_m \int_{-\infty}^{\infty} k^*(t) e^{(r_n + r_m)t} dt \\ &= \sum_{n,m=n_0}^{\infty} R_n R_m \cosh((r_n + r_m)T) K^*(ir_n + ir_m). \end{aligned}$$

By the definition (121) of  $K^*$  we have

$$K^*(ir_n + ir_m) = \frac{4\pi T}{\pi^2 + 4T^2(r_n + r_m)^2} \leq \frac{4T}{\pi}.$$

In addition, since the sequence  $\{r_n\}$  is bounded we have

$$\cosh((r_n + r_m)T) \leq e^{2T \sup |r_n|} \quad \forall n, m \in \mathbb{N}.$$

Consequently,

$$\int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} R_n e^{r_n t} \right|^2 dt \leq \frac{4T}{\pi} e^{2T \sup |r_n|} \sum_{n,m=n_0}^{\infty} |R_n| |R_m|.$$

Since  $n_0 \geq n'$ , by (80) we have that

$$\int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} R_n e^{r_n t} \right|^2 dt \leq \frac{4T}{\pi} e^{2T \sup |r_n|} \sum_{n,m=n_0}^{\infty} \frac{(|C_n|^2 + |d_n D_n|^2)^{1/2}}{m^\nu} \frac{(|C_m|^2 + |d_m D_m|^2)^{1/2}}{n^\nu}.$$

Moreover

$$\begin{aligned}
& \sum_{n,m=n_0}^{\infty} \frac{\left(|C_n|^2 + |d_n D_n|^2\right)^{1/2}}{m^\nu} \frac{\left(|C_m|^2 + |d_m D_m|^2\right)^{1/2}}{n^\nu} \\
& \leq \frac{1}{2} \sum_{m=n_0}^{\infty} \frac{1}{m^{2\nu}} \sum_{n=n_0}^{\infty} \left(|C_n|^2 + |d_n D_n|^2\right) + \frac{1}{2} \sum_{n=n_0}^{\infty} \frac{1}{n^{2\nu}} \sum_{m=n_0}^{\infty} \left(|C_n|^2 + |d_n D_n|^2\right) \\
& = \sum_{n=1}^{\infty} \frac{1}{n^{2\nu}} \sum_{n=n_0}^{\infty} \left(|C_n|^2 + |d_n D_n|^2\right).
\end{aligned}$$

Combining these inequalities we conclude that (128) is proved.  $\square$

**Theorem 5.14** *For any  $\varepsilon \in (0, 1)$  and  $T > \frac{\pi}{\gamma\sqrt{1-\varepsilon}}$  there exist  $n_0 = n_0(\varepsilon) \in \mathbb{N}$  and  $c(T) > 0$  such that*

$$\begin{aligned}
& \int_{-T}^T \left| \sum_{n=n_0}^{\infty} C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 dt \\
& + \int_{-T}^T \left| \sum_{n=n_0}^{\infty} d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} + c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 dt \\
& \leq c(T) \sum_{n=n_0}^{\infty} \left(|C_n|^2 + |d_n D_n|^2\right). \quad (129)
\end{aligned}$$

*Proof.* Since the function  $k^*(t)$  is positive, for  $n_0 \in \mathbb{N}$  to be chosen later we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 dt \\
& \leq 4 \int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} \right|^2 dt + 4 \int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} R_n e^{r_n t} \right|^2 dt \\
& + 4 \int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 dt.
\end{aligned}$$

We can apply Proposition 5.12 to the first term and to the third one and Proposition 5.13 to the second term. Therefore, fixed  $\varepsilon \in (0, 1)$  and  $T > \frac{\pi}{\gamma\sqrt{1-\varepsilon}}$  there exists  $n_0 = n_0(\varepsilon) \in \mathbb{N}$  such that, thanks to inequalities (126)–(128) and in view also of (79), we get

$$\begin{aligned}
& \int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right|^2 dt \\
& \leq c(T) \sum_{n=n_0}^{\infty} \left(|C_n|^2 + |d_n D_n|^2\right). \quad (130)
\end{aligned}$$

Moreover, in a similar way applying again Proposition 5.12 and taking into account (79) we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} k^*(t) \left| \sum_{n=n_0}^{\infty} d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} + c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right|^2 dt \\
& \leq c(T) \sum_{n=n_0}^{\infty} \left(|d_n D_n|^2 + |C_n|^2\right).
\end{aligned}$$

Combining (130) with the above inequality and recalling the notation (106) yields

$$\int_{-\infty}^{\infty} k^*(t) (|u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2) dt \leq c(T) \sum_{n=n_0}^{\infty} (|C_n|^2 + |d_n D_n|^2).$$

Now, we can consider the last inequality with the function  $k^*$  replaced by the analogous one relative to  $2T$  instead of  $T$ . So, taking into account (120), we get

$$\int_{-2T}^{2T} \cos \frac{\pi t}{4T} (|u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2) dt \leq c(2T) \sum_{n=n_0}^{\infty} (|C_n|^2 + |d_n D_n|^2),$$

whence, thanks to  $\cos \frac{\pi t}{4T} \geq \frac{1}{\sqrt{2}}$  for  $|t| \leq T$ , it follows

$$\int_{-T}^T (|u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2) dt \leq \sqrt{2} c(2T) \sum_{n=n_0}^{\infty} (|C_n|^2 + |d_n D_n|^2).$$

This completes the proof.  $\square$

Based on the approach performed in [7], the next result states that we can recover the finite number of missing terms in the inverse and direct estimates. We omit the proof, because it may be proved in much the same way as Proposition 5.8 and Proposition 5.20 of [24]. We advise the reader to keep in mind formulas (76) and (106).

**Proposition 5.15** *Let  $\{\omega_n\}_{n \in \mathbb{N}}$ ,  $\{r_n\}_{n \in \mathbb{N}}$  and  $\{\zeta_n\}_{n \in \mathbb{N}}$  be sequences of pairwise distinct numbers such that  $\omega_n \neq \zeta_m$ ,  $\omega_n \neq \overline{\zeta_m}$ ,  $r_n \neq i\omega_m$ ,  $r_n \neq i\zeta_m$ ,  $r_n \neq -\eta$ ,  $\zeta_n \neq 0$ , for any  $n, m \in \mathbb{N}$ , and*

$$\lim_{n \rightarrow \infty} |\omega_n| = \lim_{n \rightarrow \infty} |\zeta_n| = +\infty. \quad (131)$$

Assume that there exists  $n_0 \in \mathbb{N}$  such that

$$\int_0^T (|u_1^{n_0}(t)|^2 + |u_2^{n_0}(t)|^2) dt \asymp \sum_{n=n_0}^{\infty} (|C_n|^2 + |d_n D_n|^2).$$

Then, for any sequences  $\{C_n\}$ ,  $\{R_n\}$ ,  $\{D_n\}$  and  $\mathcal{E} \in \mathbb{R}$  we have

$$\int_0^T (|u_1(t)|^2 + |u_2(t)|^2) dt \asymp \sum_{n=1}^{\infty} (|C_n|^2 + |d_n D_n|^2) + |\mathcal{E}|^2. \quad (132)$$

## 5.5 Inverse and direct inequalities

We recall that

$$\begin{aligned} u_1(t) &= \sum_{n=1}^{\infty} \left( C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t} + D_n e^{i\zeta_n t} + \overline{D_n} e^{-i\overline{\zeta_n} t} \right), \\ u_2(t) &= \sum_{n=1}^{\infty} \left( d_n D_n e^{i\zeta_n t} + \overline{d_n D_n} e^{-i\overline{\zeta_n} t} + c_n C_n e^{i\omega_n t} + \overline{c_n C_n} e^{-i\overline{\omega_n} t} \right) + \mathcal{E} e^{-\eta t}, \end{aligned}$$

where

$$|\mathcal{E}|^2 \leq M \sum_{n=1}^{\infty} (|C_n|^2 + |d_n D_n|^2), \quad (M > 0). \quad (133)$$

**Theorem 5.16** *Let  $\{\omega_n\}_{n \in \mathbb{N}}$ ,  $\{r_n\}_{n \in \mathbb{N}}$  and  $\{\zeta_n\}_{n \in \mathbb{N}}$  be sequences of pairwise distinct numbers such that  $\omega_n \neq \zeta_m$ ,  $\omega_n \neq \overline{\zeta_m}$ ,  $r_n \neq i\omega_m$ ,  $r_n \neq i\zeta_m$ ,  $r_n \neq -\eta$ ,  $\zeta_n \neq 0$ , for any  $n, m \in \mathbb{N}$ . Assume that there exist  $\gamma > 0$ ,  $\alpha, \chi \in \mathbb{R}$ ,  $n' \in \mathbb{N}$ ,  $\mu > 0$ ,  $\nu > 1/2$ , such that*

$$\liminf_{n \rightarrow \infty} (\Re \omega_{n+1} - \Re \omega_n) = \liminf_{n \rightarrow \infty} (\Re \zeta_{n+1} - \Re \zeta_n) = \gamma,$$

$$\lim_{n \rightarrow \infty} \Im \omega_n = \alpha > 0,$$

$$\lim_{n \rightarrow \infty} r_n = \chi < 0,$$

$$\lim_{n \rightarrow \infty} \Im \zeta_n = 0,$$

$$|d_n| \asymp |\zeta_n|, \quad |c_n| \leq \frac{M}{|\omega_n|},$$

$$|R_n| \leq \frac{\mu}{n^\nu} \left( |C_n|^2 + |d_n D_n|^2 \right)^{1/2} \quad \forall n \geq n', \quad |R_n| \leq \mu \left( |C_n|^2 + |d_n D_n|^2 \right)^{1/2} \quad \forall n \leq n'.$$

Then, for  $\gamma > 4\alpha$  and  $T > \frac{2\pi}{\sqrt{\gamma^2 - 16\alpha^2}}$  we have

$$\int_0^T (|u_1(t)|^2 + |u_2(t)|^2) dt \asymp \sum_{n=1}^{\infty} \left( |C_n|^2 + |d_n D_n|^2 \right). \quad (134)$$

*Proof.* Since  $T > \frac{2\pi}{\sqrt{\gamma^2 - 16\alpha^2}}$ , there exists  $0 < \varepsilon < 1$  such that  $T > \frac{2\pi}{\sqrt{\gamma^2(1-\varepsilon) - 16\alpha^2(1+\varepsilon)}}$ . Therefore, thanks to Theorems 5.9 and 5.14 we are able to employ Proposition 5.15 obtaining

$$\int_0^T (|u_1(t)|^2 + |u_2(t)|^2) dt \asymp \sum_{n=1}^{\infty} \left( |C_n|^2 + |d_n D_n|^2 \right) + |\mathcal{E}|^2.$$

Finally, by (133) we can get rid of the term  $|\mathcal{E}|^2$  in the previous estimates, and hence the proof is complete.  $\square$

If we assume the condition  $|C_n| \leq M|d_n D_n|$  on the coefficients of the series instead of  $\gamma > 4\alpha$ , then we can make use of Theorem 5.10 instead of Theorem 5.9, obtaining the observability inequalities with a better estimate for the control time:  $T > \frac{2\pi}{\gamma}$ . Precisely, the following result holds.

**Theorem 5.17** *Let assume the hypotheses of Theorem 5.16 and the condition*

$$|C_n| \leq M|d_n D_n|. \quad (135)$$

Then, for  $T > \frac{2\pi}{\gamma}$  we have

$$\int_0^T (|u_1(t)|^2 + |u_2(t)|^2) dt \asymp \sum_{n=1}^{\infty} \left( |C_n|^2 + |d_n D_n|^2 \right). \quad (136)$$

## 6 Reachability results

This section will be devoted to the proof of some reachability results for wave-wave coupled systems with a memory term.

**Theorem 6.1** Let  $\beta < 1/2$ . For any  $T > \frac{2\pi}{\sqrt{1-4\beta^2}}$  and  $(u_i^0, u_i^1) \in L^2(0, \pi) \times H^{-1}(0, \pi)$ ,  $i = 1, 2$ , there exist  $g_i \in L^2(0, T)$ ,  $i = 1, 2$ , such that the weak solution  $(u_1, u_2)$  of system

$$\begin{cases} u_{1tt}(t, x) - u_{1xx}(t, x) + \beta \int_0^t e^{-\eta(t-s)} u_{1xx}(s, x) ds + au_2(t, x) = 0, \\ u_{2tt}(t, x) - u_{2xx}(t, x) + bu_1(t, x) = 0, \end{cases} \quad t \in (0, T), \quad x \in (0, \pi) \quad (137)$$

with boundary conditions

$$u_1(t, 0) = u_2(t, 0) = 0, \quad u_1(t, \pi) = g_1(t), \quad u_2(t, \pi) = g_2(t) \quad t \in (0, T), \quad (138)$$

and null initial values

$$u_i(0, x) = u_{it}(0, x) = 0 \quad x \in (0, \pi), \quad i = 1, 2, \quad (139)$$

verifies the final conditions

$$u_i(T, x) = u_i^0(x), \quad u_{it}(T, x) = u_i^1(x), \quad x \in (0, \pi), \quad i = 1, 2. \quad (140)$$

*Proof.* To prove our statement, we will apply the Hilbert Uniqueness Method described in Section 3. Let  $H = L^2(0, \pi)$  be endowed with the usual scalar product and norm

$$\|u\|_{L^2} := \left( \int_0^\pi |u(x)|^2 dx \right)^{1/2} \quad u \in L^2(0, \pi).$$

We consider the operator  $L : D(L) \subset H \rightarrow H$  defined by  $Lu = -u_{xx}$  for  $u \in D(L) := H^2(0, \pi) \cap H_0^1(0, \pi)$ . It is well known that  $L$  is a self-adjoint positive operator on  $H$  with dense domain  $D(L)$  and

$$D(\sqrt{L}) = H_0^1(0, \pi).$$

Moreover,  $\{n^2\}_{n \geq 1}$  is the sequence of eigenvalues for  $L$  and  $\{\sin(nx)\}_{n \geq 1}$  is the sequence of the corresponding eigenvectors. We can apply our spectral analysis, see Section 4.1, to the adjoint system of (137) given by

$$\begin{cases} z_{1tt}(t, x) - z_{1xx}(t, x) + \int_t^T k(s-t) z_{1xx}(s, x) ds + bz_2(t, x) = 0, \\ z_{2tt}(t, x) - z_{2xx}(t, x) + az_1(t, x) = 0, \\ z_i(t, 0) = z_i(t, \pi) = 0 \quad t \in [0, T], \\ z_i(T, \cdot) = z_i^0, \quad z_{it}(T, \cdot) = z_i^1, \end{cases} \quad t \in (0, T), \quad x \in (0, \pi) \quad i = 1, 2, \quad (141)$$

where the final data exhibit the following expansion in the basis  $\{\sin(nx)\}_{n \geq 1}$

$$z_i^0(x) = \sum_{n=1}^{\infty} \alpha_{in} \sin(nx), \quad z_i^1(x) = \sum_{n=1}^{\infty} \rho_{in} \sin(nx), \quad i = 1, 2.$$

If we take  $(z_i^0, z_i^1) \in H_0^1(0, \pi) \times L^2(0, \pi)$ ,  $i = 1, 2$ , then one has

$$\|z_i^0\|_{H_0^1}^2 = \sum_{n=1}^{\infty} \alpha_{in}^2 n^2, \quad \|z_i^1\|_{L^2}^2 = \sum_{n=1}^{\infty} \rho_{in}^2, \quad i = 1, 2. \quad (142)$$

The backward system (141) is equivalent to the forward system

$$\begin{cases} u_{1tt}(t, x) - u_{1xx}(t, x) + \int_0^t k(t-s)u_{1xx}(s, x)ds + bu_2(t, x) = 0, \\ \quad \quad \quad t \in (0, T), \quad x \in (0, \pi) \\ u_{2tt}(t, x) - u_{2xx}(t, x) + au_1(t, x) = 0, \\ u_i(t, 0) = u_i(t, \pi) = 0 \quad t \in [0, T], \\ \quad \quad \quad i = 1, 2, \\ u_i(0, \cdot) = z_i^0, \quad u_{it}(0, \cdot) = z_i^1, \end{cases} \quad (143)$$

that is, if  $(u_1, u_2)$  is the solution of (143), then the solution  $(z_1, z_2)$  of (141) is given by

$$z_1(t, x) = u_1(T-t, x), \quad z_2(t, x) = u_2(T-t, x).$$

Therefore, thanks to the representation for the solution of (143), see Theorem 4.5, we can write  $(z_1, z_2)$  in the following way, for any  $(t, x) \in [0, T] \times [0, \pi]$

$$\begin{aligned} z_1(t, x) &= \sum_{n=1}^{\infty} \left( C_n e^{i\omega_n(T-t)} + \overline{C_n} e^{-i\overline{\omega_n}(T-t)} + R_n e^{r_n(T-t)} + D_n e^{i\zeta_n(T-t)} + \overline{D_n} e^{-i\overline{\zeta_n}(T-t)} \right) \sin(nx), \\ z_2(t, x) &= \sum_{n=1}^{\infty} \left( d_n D_n e^{i\zeta_n(T-t)} + \overline{d_n D_n} e^{-i\overline{\zeta_n}(T-t)} + c_n C_n e^{i\omega_n(T-t)} + \overline{c_n C_n} e^{-i\overline{\omega_n}(T-t)} \right) \sin(nx) \\ &\quad + e^{-\eta(T-t)} \sum_{n=1}^{\infty} E_n \sin(nx). \end{aligned}$$

In particular, thanks also to (142) we get

$$\sum_{n=1}^{\infty} n^2 \left( |C_n|^2 + |d_n D_n|^2 \right) \asymp \|z_1^0\|_{H_0^1}^2 + \|z_1^1\|_{L^2}^2 + \|z_2^0\|_{H_0^1}^2 + \|z_2^1\|_{L^2}^2. \quad (144)$$

Moreover, for any  $t \in [0, T]$

$$\begin{aligned} z_{1x}(t, \pi) &= \sum_{n=1}^{\infty} (-1)^n n \left( C_n e^{i\omega_n(T-t)} + \overline{C_n} e^{-i\overline{\omega_n}(T-t)} + R_n e^{r_n(T-t)} + D_n e^{i\zeta_n(T-t)} + \overline{D_n} e^{-i\overline{\zeta_n}(T-t)} \right), \\ z_{2x}(t, \pi) &= \sum_{n=1}^{\infty} (-1)^n n \left( d_n D_n e^{i\zeta_n(T-t)} + \overline{d_n D_n} e^{-i\overline{\zeta_n}(T-t)} + c_n C_n e^{i\omega_n(T-t)} + \overline{c_n C_n} e^{-i\overline{\omega_n}(T-t)} \right) \\ &\quad + e^{-\eta(T-t)} \sum_{n=1}^{\infty} (-1)^n n E_n. \end{aligned}$$

We can apply Theorem 5.16 to  $(z_{1x}(t, \pi), z_{2x}(t, \pi))$ . Indeed, thanks to the above expressions for  $z_{ix}(t, \pi)$ ,  $i = 1, 2$ , and (134) we have

$$\int_0^T (|z_{1x}(t, \pi)|^2 + |z_{2x}(t, \pi)|^2) dt \asymp \sum_{n=1}^{\infty} n^2 \left( |C_n|^2 + |d_n D_n|^2 \right),$$

and hence by (144) we get

$$\int_0^T (|z_{1x}(t, \pi)|^2 + |z_{2x}(t, \pi)|^2) dt \asymp \|z_1^0\|_{H_0^1}^2 + \|z_1^1\|_{L^2}^2 + \|z_2^0\|_{H_0^1}^2 + \|z_2^1\|_{L^2}^2. \quad (145)$$

Therefore, we have proved Theorem 3.1. Furthermore, we consider the linear operator  $\Psi$  introduced in Section 3 and, thanks to (33), defined by

$$\Psi(z_1^0, z_1^1, z_2^0, z_2^1) = (-u_{1t}(T, \cdot), u_1(T, \cdot), -u_{2t}(T, \cdot), u_2(T, \cdot)),$$

where  $(u_1, u_2)$  is the weak solution of system (137). We have that

$$\Psi : H_0^1(0, \pi) \times L^2(0, \pi) \times H_0^1(0, \pi) \times L^2(0, \pi) \rightarrow H^{-1}(0, \pi) \times L^2(0, \pi) \times H^{-1}(0, \pi) \times L^2(0, \pi)$$

is an isomorphism. Therefore, for  $(u_i^0, u_i^1) \in L^2(0, \pi) \times H^{-1}(0, \pi)$ ,  $i = 1, 2$ , there exists one and only one  $(z_1^0, z_1^1, z_2^0, z_2^1) \in H_0^1(0, \pi) \times L^2(0, \pi) \times H_0^1(0, \pi) \times L^2(0, \pi)$  such that

$$\Psi(z_1^0, z_1^1, z_2^0, z_2^1) = (-u_1^1, u_1^0, -u_2^1, u_2^0).$$

Finally, if we consider the solution  $(z_1, z_2)$  of system (141) with final data given by the unique  $(z_1^0, z_1^1, z_2^0, z_2^1)$ , then the control functions required by the statement are given by

$$g_1(t) = z_{1x}(t, \pi) - \beta \int_t^T e^{-\eta(s-t)} z_{1x}(s, \pi) ds, \quad g_2(t) = z_{2x}(t, \pi),$$

that is, our proof is complete.  $\square$

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